An Optimal Algorithm for Roundness Determination on Convex Polygons

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Abstract

In tolerancing, the Out-Of-Roundness factor determines the relative circularity of planar shapes. The measurement of concern in this work is the Minimum Radial Separation, as recommended by the American National Standards Institute (ANSI). Here we show that the algorithm given in Le and Lee [4] runs in $\Theta(n^2)$ time even for convex polygons. Furthermore, we present an optimal $O(n)$ time algorithm to compute the Minimum Radial Separation of convex polygons, which represents not only a factor n improvement over the previously best known algorithm, but also a factor of $\log n$ improvement over Le and Lee's conjectured complexity for the problem.

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†Supported in part by the National Science Foundation under the Grant No. CCR-8901815.

1 Introduction

A vital control in the physical processes of production and manufacturing is the *tolerance factor*. In order to obtain a comparative measurement in qualifying adherence to design specifications, tolerance yields a relative figure for any given attribute.

Physical shape is one of many such attributes. This work is concerned with the tolerance factor known as the *Out-of-Roundness* factor, which determines the extent to which a given planar shape deviates from a circle. The Out-of-Roundness factor is determined by using one of several metrics, of which, in practice, four are commonly used: the *Least Squares Center* (LSC), the *Maximum Inscribed Center* (MIC), the *Minimum Circumscribed Center* (MCC), and the *Minimum Radial Separation* (MRS) center [7]. The LSC center designates a circle from which the sum of the squares of radial ordinates in a polar profile of the planar shape is minimized. The MIC center designates a circle of maximal radius which is inscribed, or completely contained, within the planar shape. The MCC center designates a circle of minimal radius which is circumscribed around, or completely contains, the planar shape. The MRS center designates two concentric circles, one which is circumscribed, the other inscribed, having minimal radial difference. The American National Standards Institute (ANSI Standard B89.3.1-1972), recommends the use of the MRS center to measure Out-of-Roundness. This work concerns the MRS center.

Until recently, no exact method existed for finding the MRS center. There have been, however, several heuristical methods which would yield a general approximation to the true MRS center. Recently, the problem was solved for input in terms of a point set [2], but that result cannot be extended to simple polygons or more general planar shapes, as the MRS center, as calculated, could be found to lie outside the original boundary of a planar shape. More recently, Le and Lee [4] solved the MRS problem for input in terms of simple polygons, through methods of computational geometry. That solution relies upon intersecting the medial axis and the farthest neighbor Voronoi diagram of the polygon, and requires $O(n \log n + k)$ time, where n is the number of vertices, and k is the number of intersections between these two structures.

This work, as seen in Section 3, further clarifies the complexity, showing that the number of intersections, k, can be $\Theta(n^2)$, even when restricted to convex polygons. In Section 4.1, a new algorithm is presented that solves the MRS problem on convex polygons, in $O(n)$ time. This is an improvement of a factor of n over the previously best known algorithm, as well as a factor of $\log n$ better than a conjecture of Le and Lee [4]. This algorithm also makes use of the medial axis and farthest neighbor Voronoi diagram, but does not require their intersection, thereby yielding the improvement in complexity.

2 Definitions

Definition 1 An *inscribed circle*, $IC_G(C, R)$, is a circle inscribed in a simple polygon G, having a center C, internal to G, with a radius R.

Definition 2 A *circumscribed circle*, $CC_G(C, R)$, is a circle circumscribed around a simple polygon G, having a center C, internal to G, with a radius R.

Definition 3 The *convex hull*, $CH(P)$, of a point set $P = \{p_1, p_2, \ldots, p_n\}$, is the smallest convex polygon completely containing P.

Definition 4 The *Minimal Radial Separation center*, MRS, of a simple polygon G, is defined as any point, c, internal to G, being the center of of two concentric circles, $IC_G(c, r_1)$, and $CC_G(c, r_2)$, such that the radial separation of these two circles is minimal for all points internal to G. Note that there exists at least one such point, and possibly several, in the case of non-convex polygons.

Definition 5 The *medial axis*, $MA(G)$, of a simple polygon G, is the set of all points internal to G, for which each point is closest to at least two points on the boundary of G. The medial axis, also known as the skeleton or symmetric axis, is a tree-like planar graph comprised of straight line segments and portions of parabolic curves [6]. The parabolic segments contain points which are closest to one *reflex*[1] vertex of G, and one segment of the boundary of G. Thus in the case of convex polygons, the medial axis is simply comprised of straight line segments. The medial axis divides the polygon into regions whose interiors are closer to either one boundary segment, or one reflex vertex, than any other section of the boundary.

Definition 6 The *farthest neighbor Voronoi diagram*, $FNV(P)$, of a point set P, is a division the plane into (unbounded) regions which contain all such points that are farther from one specific point in P than any other.

[1]A reflex vertex of a polygon has an internal angle greater than π.

The diagram consists of all points on the plane having equivalent maximal distances to more than one point in P. A point in P has a non-empty region in $FNV(P)$ if and only if that point lies on $CH(P)$ [9, 5].

Definition 7 An *arc bisector*, B_c, of a circular arc segment ARC_c, centered at c, passes through the point on the arc which partitions the arc segment into two halves of equal arc length.

3 An Improved Analysis of Le & Lee's Algorithm

Le and Lee have shown that while the MRS center of a simple polygon G may not be unique, at least one MRS center must lie on the the medial axis, $MA(G)$ [4]. Thus

$$|IC_G(C_{MRS}, R) \cap G| \geq 2,$$

while

$$|CC_G(C_{MRS}, R) \cap G| \geq 1,$$

or that the inscribed circle centered at the MRS center is adjacent to at least two points on the boundary of G, but that the respective circumscribed circle is only adjacent to at least one point on the boundary of G. This leads to an $O(n \log n + k)$ algorithm of Le and Lee [4] which relies on the k intersections between the medial axis and the farthest neighbor Voronoi diagram.

A tight upper bound on the number of intersections between the medial axis and the farthest neighbor Voronoi diagram of a simple polygon has remained an open problem. Trivially, an upper bound is $O(n^2)$, since both of these structures are comprised of $O(n)$ segments, and the number of intersections between subsegments are constant. As shown below, this bound is found to be tight, even when restricted to convex polygons.

Theorem 1 *There exist polygons for which the maximum number of intersections between the medial axis, and the farthest neighbor Voronoi diagram, is* $\Theta(n^2)$.

Proof Trivially, we know that the number of intersections cannot be larger than $O(n^2)$, as discussed above. To show that this bound is indeed tight, consider the following example of a convex polygon, G, on n vertices, where:

- $n/2$ vertices are distributed evenly around a half circle, ARC_{C_r}, having an arc bisector B_r.

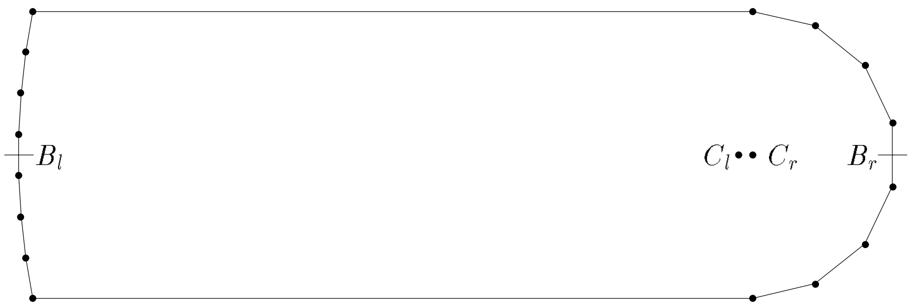

Figure 1: A convex polygon having $\Theta(n^2)$ FNV-MA intersections.

- $n/2$ vertices are distributed evenly around a second circular arc segment, ARC_{C_l}, having an arc bisector B_l, of relatively large radius and small arc length.
- The two arcs are sized and positioned such that:
  - The four points, B_l, C_l, C_r, B_r, are collinear, and appear in the stated order.
  - C_l, and C_r are separated by a very small ϵ–distance.
  - ARC_{C_l} has sufficiently large radius, such that the segments of $FNV(G)$ representing the vertices of ARC_{C_l} pass through ARC_{C_r} at most a small constant number of vertices away from B_r.

Such a polygon is shown in Figure 1, and with greater detail in Figure 2.

Thus, each of the $n/2$ $FNV(G)$ segments related to the vertices of ARC_{C_l} originate at C_l and extend through the polygon within a constant number of vertices away from B_r, on ARC_{C_r}. The $n/2$ segments of $MA(G)$ corresponding to the vertices on ARC_{C_r} extend from each vertex and meet at C_r. Thus at least $n/2-c$ segments of $FNV(G)$ intersect with at least $n/4-d$ segments of $MA(G)$, for some constants c, d. Therefore, the number of intersections between the medial axis, and the farthest neighbor Voronoi diagram, of this convex polygon, is $\Theta(n^2)$. □

Through application of Theorem 1, the following corollary refines the complexity of Le and Lee's algortihm [4].

Corollary 1 *The worst case complexity of the MRS algorithm presented*

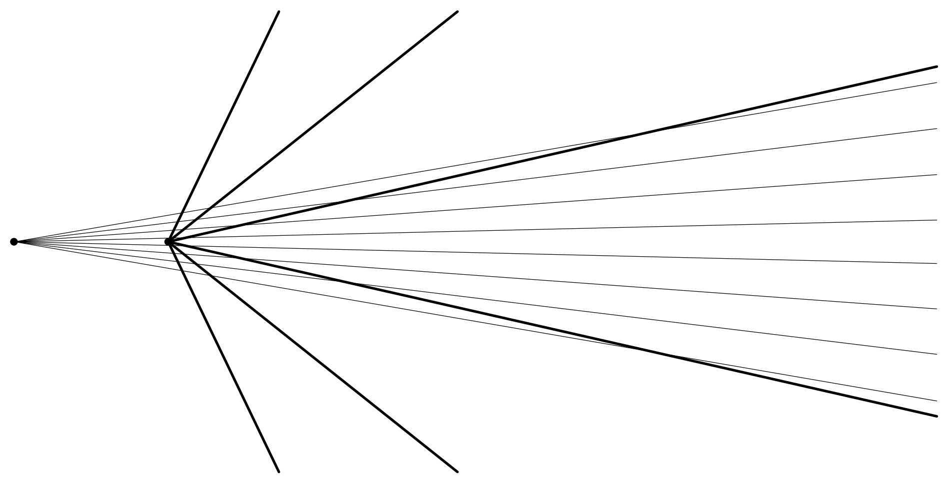

Figure 2: Detail of Fig. 1, showing important FNV and MA intersection points.

in [4] is $\Theta(n^2)$, dominated by the intersection of medial axis and farthest neighbor Voronoi diagram.

4 A Linear Time MRS Algorithm on Convex Polygons

To facilitate an efficient MRS algorithm for convex polygons, certain properties are first identified. It has been proven that for convex polygons, an MRS center is always found on an intersection point between $FNV(G)$ and $MA(G)$ [4]. A version of which follows:

Lemma 1 *The MRS center of a convex polygon always lies on an intersection point between the medial axis, and the farthest neighbor Voronoi diagram of the polygon.*

To be able to find an MRS center on convex polygons in optimal (or linear) time implies that the FNV-MA intersection having minimum RS value can be found without enumerating all such intersections. Thus, in investigating the RS value in relation to these structures, the atomic element is found to be a division of the segments of the medial axis by the farthest neighbor, or *medial axis subsegments* (MASS). That is to say, a decomposition of the medial axis into its constituent line segments, and for each such segment that intersects with the farthest neighbor Voronoi diagram, a subdivision into non-intersecting line segments. Thus each MASS comprises the unique set of points equidistant to one closest pair of polygon edges, and farthest from one specific vertex of the polygon. The following properties are thereby identified:

Lemma 2 *For any two distinct points, p and q, on any MASS in a convex polygon G, closest to points r and s, respectively, on the boundary of G, and farthest from vertex f, radial separation at point p is less than at q, if and only if f is closer to the line $|pr|$ than $|qs|$.*

Proof Since the nearest neighbor distance function is functionally symmetric to two polygon edges, it is assumed that f, r, and s all lie on the same side of the line determined by pq. Because G is convex, the farthest neighbor cannot lie in the infinite trapezoidal (or triangular) region formed by the segment pq, and two rays extending from p and q through r and s, respectively. Furthermore, it cannot lie on the opposite side of the boundary segment rs (Figure 3). This region, $pqsr$, is henceforth called the MASS Proprietary Region (MPR). Let a, b, c, d, and e represent the distances $|fp|$, $|fq|$, $|pr|$, $|qs|$, and $|qr|$, respectively. Consider the quadrilateral $fpqr$. By plane geometry,

$$\begin{aligned} b+c &> a+e \\ a-c &< b-e \\ b-e &\leq b-d \\ a-c &< b-d \end{aligned}$$

Thus, radial separation at p is less than at q. If f lies closer to the line $|qs|$ than $|pr|$, then, symmetrically, radial separation at q is less than at p. □

Corollary 2 *The RS value along any MASS, in a convex polygon, is strictly increasing or strictly decreasing.*

Lemma 3 *If any MASS, with endpoints A and B, where RS is increasing from A to B, is connected to another MASS with endpoints B and C, then RS is increasing from B to C.*

Proof Given two such MASSes, one of the following cases applies:

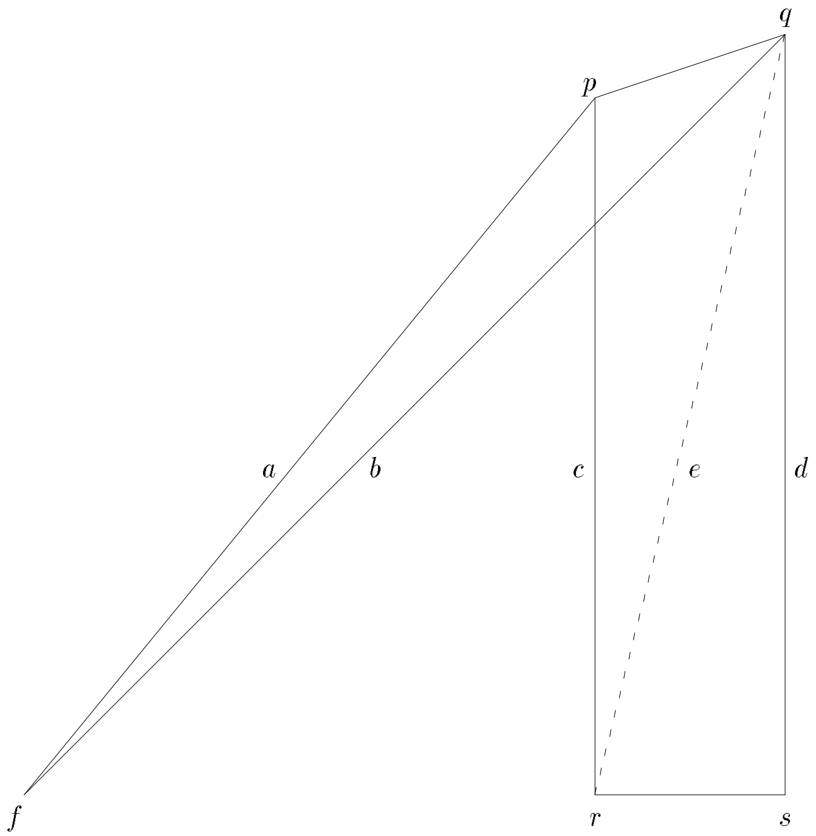

Figure 3: Illustration of Lemma 2.

Case 1: The connected MASSes are both part of the same MA segment, but have different farthest vertices. Let p and q be the farthest neighbors from AB and BC, respectively. In this case, both farthest neighbors must lie on a circle centered at B. The vertex p must also lie on a smaller circle centered at A, since RS is increasing from A to B. This circle must contain q. Thus q must lie on an arc on the circle centered at B interior to the circle centered at A. Since RS is increasing from A to B, q must lie according to the same constraints of Lemma 2, and thus RS is increasing from B to C. (Figure 4).

Case 2: The connected MASSes are both farthest from the same vertex, but are comprised from different MA segments. The adjacent MASS cannot lie within the MPR of the given MASS. By Lemma 2, the farthest neighbor lies closer to the line connecting A to its projection onto the nearest neighbor segment, than to that of B. Thus, the farthest neighbor is closer to the line connecting B to its nearest neighbor projection, than to that of C. Thus, RS is increasing from B to C.

Case 3: The connected MASSes have both different farthest vertices, and are comprised from different MA segments. This occurs when the FNV diagram intersects the MA where two MA segments meet. Let p and q be the farthest neighbors from AB and BC, respectively. Since RS is increasing from A to B, p and BC must lie on opposite sides of both MPR's of MASS AB. The vertex q must lie on the circle centered at B on which p lies. More specifically, it must lie on the arc segment from p to the point symmetric to p with respect to the line determined by MASS segment AB (Figure 5). Thus q and BC must lie on opposite sides of both MPR's of MASS AB. Therefore both MPR regions of MASS BC also lie opposite of q, and thus the distance from q to either MPR of MASS BC is decreasing when moving from B to C. Therefore, by Lemma 2, RS must be increasing from B to C.

□

Lemma 4 *The RS is strictly increasing when proceeding along the medial axis, outwards from some intersection of the medial axis and the farthest neighbor Voronoi diagram.*

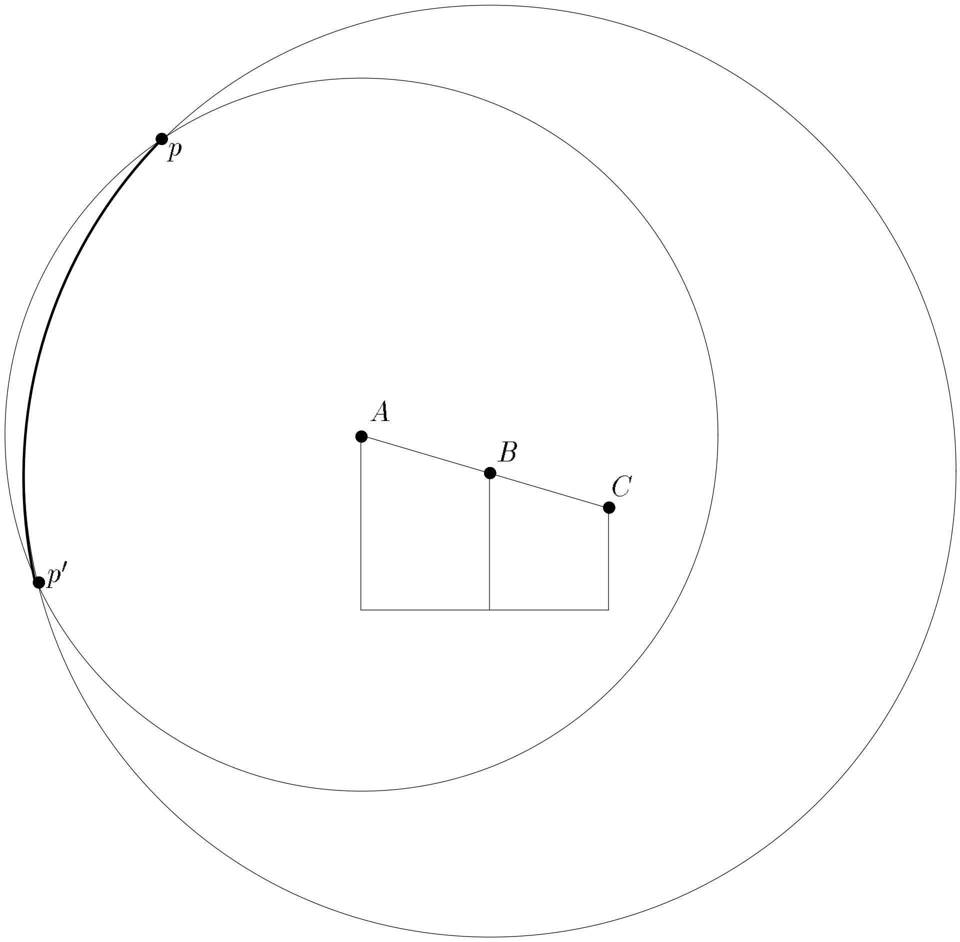

Figure 4: Case 1: The vertex q must lie on the arc pp'.

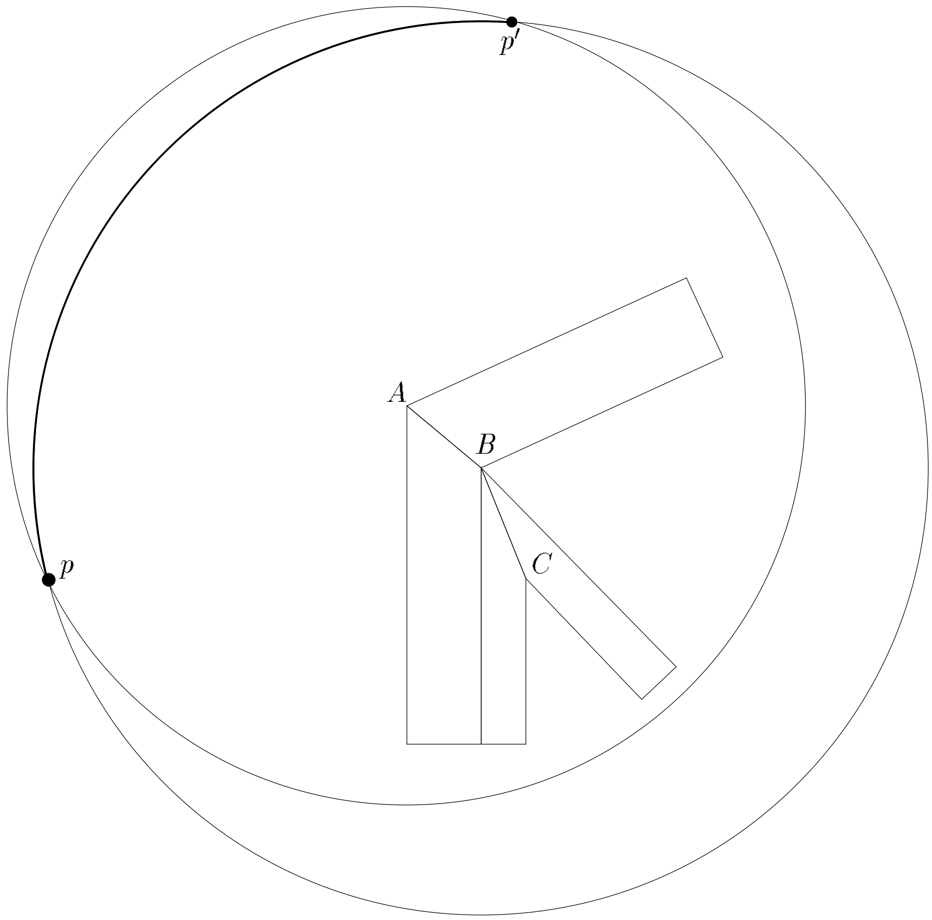

Figure 5: Case 3: The vertex q must lie on the arc pp'.

Proof Since RS is minimal at the MRS, which is at an intersection of the medial axis and the farthest neighbor Voronoi diagram. Therefore the MRS is found at the endpoint of at least one MASS. Consider the division of the medial axis into MASSes as a tree structure rooted at the MRS. Since all lower level MASS segments are connected at the maximal RS point of the segment at the next higher level, RS value is strictly increasing when proceeding from the MRS to any leaf node. □

Corollary 3 *The MRS center on convex polygons is unique.*

4.1 The Algorithm

The following algorithm is first presented in order to facilitate efficient searching within the medial axis, to find that medial axis segment containing the MRS center:

Algorithm 1 [MA-Search]

1. Perform a depth first search in the medial axis to find an edge having minimal difference between the sizes of the two subtrees it connects, amongst all edges in the medial axis tree.

2. If the ratio of sizes of the smaller subtree to the larger, at this edge, is strictly less than a predetermined constant, in the range $[0, 1/2]$, then:

    Compute the first derivative of the radial separation at both endpoints to determine which of the the subgraphs contains the MRS center. Remove the other two subgraphs from the tree. If the remaining subgraph contains more than one edge, return to step 1, otherwise stop, returning this edge.

   otherwise:

    Consider the endpoint of this edge connecting to the larger subtree. Locate this point in the farthest neighbor Voronoi diagram to determine the farthest vertex of the polygon for each incident edge. The point must lie according to one of the following situations:

No intersection: All edges are farthest from the same vertex.
Intersection with an FNV edge: Determine which half-plane each MA edge lies in.
Intersection with an FNV vertex: Merge the incident MA and FNV edges. Sweep through these edges to determine within which FNV region each MA edge lies.

Compute the first derivative of radial separation for each incident MA edge. Remove all incident MA edges and their respective subtrees, except that edge (if any) which has decreasing radial separation. If all incident edges are removed, stop, returning this point. Otherwise, return to step 1.

Analysis There are, of course, a linear number of segments in both the medial axis and the farthest neighbor Voronoi diagram of a convex polygon, since both are planar graphs (see, for example, [8]). To find an MA edge such that difference in sizes of its subtrees are minimal, a depth first search is performed on the tree, counting the number of edges found in each subtree. Next, progress through the tree, starting at any edge, comparing the separation ratio along edges. If an edge is encountered that divides the tree into two subtrees of sizes $\lfloor n/2 \rfloor - 1$, and $\lceil n/2 \rceil - 1$, the search stops there, as this is optimal. Otherwise, the search will have reached an MA vertex where all edges connect a subtree of less than $\lfloor n/2 \rfloor - 1$ edges. The edge having the largest subtree is chosen. Since every edge is visited at most twice, step 1 is $O(n)$.

Point location within the farthest neighbor Voronoi diagram can be solved in $O(\log n)$ time, using the algorithm of Edelsbrunner, et al [3]. Once found, the first derivatives of radial separation can be computed in constant time. To determine which half-plane an edge lies in, also requires constant time, or $O(n)$ to compute all incident edges. Similarly the MA and FNV edges incident on a common point can be merged and swept in $O(n)$ time. Since each step removes all but one edge, the total complexity of determining the farthest vertex of MA edges is $O(n)$. The total complexity of FNV search operations is $O(\log^2 n)$, as the algorithm performs at most a constant number of searches in each cycle, and cycles at most $O(\log n)$ times. Step 2, and the algorithm as a whole, is $O(n)$, since at least a constant fraction of medial axis edges are removed in each cycle. □

Given the above algorithm to perform searching in the medial axis in linear time, the following algorithm is derived to locate the MRS center in convex polygons.

Algorithm 2 [Find-MRS]

1. Compute the medial axis.
2. Compute the farthest neighbor Voronoi diagram.
3. Perform preprocessing necessary for search in the farthest neighbor Voronoi diagram.
4. Perform **MA-Search** to locate the medial axis segment containing the MRS center.
5. Compute the intersection of the located segment with the farthest neighbor Voronoi diagram.
6. Compute the radial separation at each intersection and return the minimum.

Analysis Linear-time algorithms exist for computing both structures on convex polygons[1]. Thus Steps 1 and 2 are both $O(n)$. In order to optimally perform point location within the farthest neighbor Voronoi diagram, O(n) preprocessing must be performed, according to the algorithm of Edelsbrunner, et al [3]. Step 3 is thus O(n). Step 4 is $O(n)$, as discussed above. The intersection of one medial axis segment and the farthest neighbor Voronoi diagram can be determined in $O(n)$ time. Step 5 is thus $O(n)$. The computation of the radial separation of all such intersections is $O(n)$. The minimum point of intersection, in terms of radial separation, is computed in $O(n)$ time. Step 6, and the algorithm as a whole, is then $O(n)$. $\square$

Theorem 2 *The MRS center on convex polygons can be determined in $O(n)$ time.*

Proof Through application of Lemma 4, Algorithm 2, as described above, locates the MRS center of a convex polygon by searching the medial axis in linear time. $\square$

5 Concluding Remarks

It has been shown that the minimum radial separation of a convex polygon can be computed in linear time, where n is the number of vertices in the polygon, through use of the medial axis, and the farthest neighbor Voronoi diagram.

Additionally, it has been shown that the worst case complexity of the algorithm of Le and Lee [4] to compute the minimum radial separation of a simple polygon is $\Theta(n^2)$. This is based upon the discovery that the number of intersections between the medial axis and the farthest neighbor Voronoi diagram of all simple polygons is $\Theta(n^2)$, in the worst case.

Finally, it should be noted that the lower bound of the MRS problem for star-shaped and other non-convex polygons remains an interesting problem.

6 Acknowledgements

The authors wish to thank Dr. Christos Levcopoulos for his helpful comments in optimizing the the complexity of the algorithm, and Dr. Arne Andersson for many helpful comments.

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