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SKEW VORONOI DIAGRAMS*

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ABSTRACT

On a tilted plane T in three-space, *skew distances* are defined as the Euclidean distance plus a multiple of the signed difference in height. Skew distances may model realistic environments more closely than the Euclidean distance. Voronoi diagrams and related problems under this kind of distances are investigated. A relationship to convex distance functions and to Euclidean Voronoi diagrams for planar circles is shown, and is exploited for a geometric analysis and a plane-sweep construction of Voronoi diagrams on T. An output-sensitive algorithm running in time $O(n \log h)$ is developed, where n and h are the numbers of sites and non-empty Voronoi regions, respectively. The all nearest neighbors problem for skew distances, which has certain features different from its Euclidean counterpart, is solved in $O(n \log n)$ time.

Keywords: Direction-dependent distance, Voronoi diagram, additive weights, output-sensitive algorithm

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1. Introduction

Most computational geometry research on planar problems assumes that the underlying plane is perfectly 'flat', in the sense that movement between any two points on the plane always takes the same cost as long as the Euclidean distance between the two points is the same. This assumption, however, may not be realistic in certain practical applications. In real environments, distances may depend on the direction one moves along,[15] or even may be influenced by local properties of the plane.[11] These situations sometimes can be modeled by considering a piecewise-linear surface as the underlying 'plane', and measuring distances therein (see e.g. Ref. [10]). In fact, many distance problems on non-flat planes are hard to deal with from the computational geometry point of view.

In this paper, we offer a study of distance problems for the basic case of a 'tilted' plane in three-space. In this model, the cost of moving depends not only on the Euclidean distance but also on how much upwards or downwards the movement has to travel, simulating the situation when driving a vehicle on the tilted plane. This gives rise to distance functions that we call the *skew distances*, which are direction-sensitive and, in particular, non-symmetric.

Perhaps the most prominent structure defined by means of distances is the Voronoi diagram (see e.g. Refs. [2,3] for survey papers on this subject). Many other distance-based structures, like nearest-neighbor graphs and cluster structures, can be computed efficiently once the Voronoi diagram is available. We thus study skew distances primarily through the Voronoi diagrams they define. Despite being a quite natural concept, direction-sensitive distances and their Voronoi diagrams have not received much attention in the computational geometry literature. Exceptions are convex distance functions[5,8] and the so-called boat-sail distance[15] which comes closest to our concept of skew distances.

Basic properties of skew distances are given in Section 2. Section 3 introduces skew Voronoi diagrams and draws a connection to Euclidean Voronoi diagrams for circles on the 'flat' plane. Bounds on the size, and a general $O(n \log n)$ time construction algorithm result from this relationship. The impact of the slope of the tilted plane on the geometry of skew Voronoi diagrams is investigated in Section 4. Section 5 presents an output-sensitive construction algorithm that runs in $O(n \log h)$ time, where h is the number of non-empty Voronoi regions. The all nearest neighbor problem for skew distances, which has certain features different from its Euclidean counterpart, is solved in $O(n \log n)$ time.

2. Tilted Plane and Skew Distances

Let T be a tilted plane in three-space $\mathcal{R}^3$ such that the angle between T and the xy-plane is α, with $0 \leq \alpha \leq \pi/2$. By convention, T is obtained by rotation along the x-axis. We define a coordinate system on T by taking the accordingly rotated coordinate system of the xy-plane. Based on this system, a point p on T is described by its coordinates $x(p)$ and $y(p)$. Note that the Euclidean distance in $\mathcal{R}^3$ between $p, q \in T$ can be expressed as $d_2(p,q) = ((x(p)-x(q))^2 + (y(p)-y(q))^2)^{1/2}$.

Thus we need not be involved with the heights of p and q in $\mathcal{R}^3$.

A simple distance function on T is obtained by taking, for points $p, q \in T$, their Euclidean distance plus their signed difference in height, that is, $d_2(p,q) + (h(q) - h(p))$. The latter term is equal to $(y(q) - y(p)) \cdot \sin\alpha$, which means that the slope of T affects the y-difference of p and q by a factor in the range $[0,1]$. To obtain a richer, but still realistic class of distance functions on T, we define the *skew distance* from p to q as

$$d(p,q) = d_2(p,q) + k \cdot d_y(p,q),$$

where $d_y(p,q) = y(q) - y(p)$, and $k \geq 0$ is a constant. For $k = 0$, $d(p,q)$ is simply the Euclidean distance, corresponding to the case of $\alpha = 0$ where T coincides with the xy-plane. The parameter k has a nice physical interpretation. Imagine a ball moving on T, and let $\frac{1}{k}$ be the frictional coefficient on T. For $k < 1$, friction dominates gravity and the ball sticks on the skew plane. Friction and gravity balance out if $k = 1$. For $k > 1$, gravity dominates friction and the ball rolls downhill. The last case also models the assumption that energy is gained when a (electric-driven) vehicle moves downhill.

The skew distance d is non-symmetric unless $k = 0$. Note also that d obeys the triangle inequality. In the case of $k > 1$, $d(p,q)$ may be negative if q lies below p. Moreover, $d(p,q)$ may *decrease* as q is moving downwards. In fact, for $k \geq 1$, there are infinitely many points at skew distance 0 from p. Let

$$L_0(p) = \{a \mid d(p,a) = 0\}.$$

Clearly, for $k < 1$, $L_0(p) = \{p\}$. For $k = 1$, $L_0(p)$ is the vertical ray emanating from p and extending below p. For $k > 1$, $L_0(p)$ is composed of two rays of slopes $\pm 1/(\sqrt{k^2-1})$, emanating from and extending below p. We refer to these two rays as the *0-rays* of p. The area below $L_0(p)$ is called the *negative area* of p, denoted by $N(p)$. Any point $a \in N(p)$ is at a negative skew distance from p, and thus is closer to p than p to itself.

As the skew distance d is non-symmetric, two different 'unit circles' with point p as the fixed center can be defined, $\sigma(p) = \{a \mid d(p,a) = 1\}$ and $\sigma'(p) = \{a \mid d(a,p) = 1\}$. Intuitively speaking, $\sigma(p)$ is the locus of points that can be reached from p at a unit cost while $\sigma'(p)$ is the locus of points that can reach p at a unit cost. Observe that $\sigma'(p)$ is just the reflection of $\sigma(p)$ through the horizontal line passing through p. Thus we keep the convention of considering only the 'outgoing' skew distance in the definitions of geometric structures, keeping in mind that their 'incoming' versions can be obtained by reflection.

Lemma 1 *For $k > 0$, $\sigma(p)$ is a conic with focus p, directrix the horizontal line at y-distance $1/k$ above p, and eccentricity k. Thus $\sigma(p)$ is an ellipse for $0 < k < 1$, a parabola for $k = 1$, and a branch of a hyperbola for $k > 1$.*

Proof. Let a be any point on $\sigma(p)$. From $d(p,a) = 1$, we get $d_2(p,a) = 1 - k \cdot d_y(p,a)$. Note that the highest point of $\sigma(p)$ thus lies at y-distance $1/(k+1)$ above p. Let now ℓ be the horizontal line at y-distance $1/k$ above p. Clearly, $\sigma(p)$ entirely lies below ℓ. Thus $d_y(p,a) = 1/k - d_2(\ell, a)$, and we conclude that

$d_2(p, a) = k \cdot d_2(\ell, a)$. In other words, for any $a \in \sigma(p)$, the Euclidean distance from p is k times its Euclidean distance from ℓ. But then a belongs to a conic with focus p, directrix ℓ, and eccentricity k, see e.g. Ref. [4]. □

3. Skew Voronoi Diagrams and Related Structures

Let S be a set of n points on the tilted plane T. We refer to the points in S as *sites*, and are interested in geometric structures that can be defined by means of S and the skew distance on T. A basic structure that reflects a lot of distance information is the Voronoi diagram of a set of sites. In our case, it associates each site $p \in S$ with the region

$$reg(p) = \{a \mid d(p, a) < d(q, a), \forall q \in S\}.$$

The collection of all (non-empty) regions defined by S, together with their bounding edges and vertices, is termed the *skew Voronoi diagram* of S, or $SV(S)$ for short. In this 'outgoing' version of diagram, sites may be viewed as 'emergency service providers' (e.g., ambulances) that respond to emergency calls (i.e., query points) by getting to the locations of emergency in the fastest possible manner. In the corresponding 'incoming' version, the sites may be 'emergency service facilities' (e.g., hospitals) to which users (i.e., query points) go at the minimum traveling time. Figures 1, 2, and 5 illustrate the skew Voronoi diagram for a fixed set of six sites and for different values of the 'slope parameter' k. In Figure 1, the dashed line segments indicate the upper envelope of the 0-rays (see Section 4.1). Note the emptiness of $reg(q)$ and $reg(t)$.

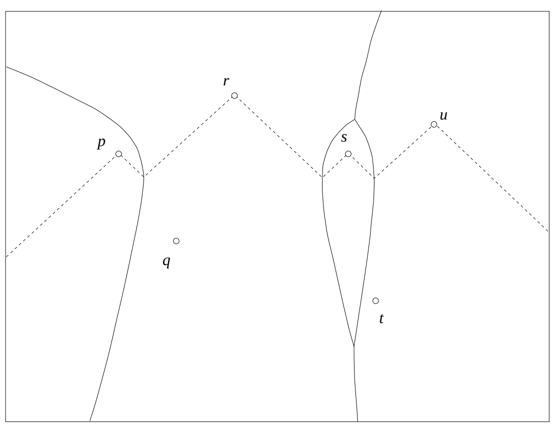

Fig. 1. Skew Voronoi diagram for $k = 1.5$

It can be seen that $SV(S)$ shows different behavior for $k < 1$, $k = 1$, and $k > 1$, respectively. The case $k < 1$ leads to known structures. By Lemma 1, the skew distance d is a *convex distance function* as defined in Ref. [5], and Voronoi diagrams for convex distance functions are well-studied objects; see e.g. Ref. [8]. In fact, for $k < 1$, $SV(S)$ has a combinatorial structure identical to the Euclidean Voronoi diagram of *some* set S' of point sites[a]: Apply the affine transformation τ that takes the elliptic circles defined by d (that all have the same eccentricity k) into Euclidean circles, and note that τ sends ellipses empty of sites in S into circles empty of sites in $S' = \tau(S)$. Note also that the convex hulls of S' and S are combinatorially the same, so exactly the sites lying on the convex hull of S have unbounded regions in $SV(S)$. Still, $SV(S)$ and the Euclidean Voronoi diagram of S are combinatorially different, in general. For example, two sites which are neighbors in the x-sorted order of S but have non-adjacent Euclidean regions will have adjacent skew regions, provided k is sufficiently close to 1; see Section 4.2.

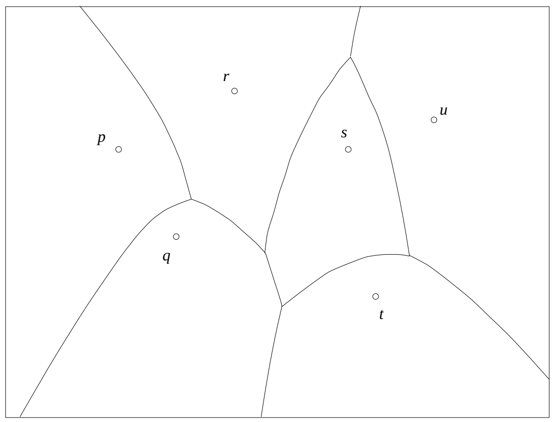

Fig. 2. Skew Voronoi diagram for $k = \frac{1}{2}$.

For $k \geq 1$, the unit circle for d fails to be bounded. It seems not clear whether the known techniques for analyzing and constructing convex distance Voronoi diagrams can be generalized. Still, the skew Voronoi diagram has another nice and helpful interpretation for general k which is described below.

The *bisector*, $b(p,q)$, of two sites $p, q \in S$ is defined as the set of points at equal skew distance from p and q. That is,

$$b(p,q) = \{a \mid d(p,a) = d(q,a)\}.$$

Note that each region of $SV(S)$ is the intersection of portions of T which are bounded by bisectors of sites. Simple calculations show that $b(p,q)$ is described by the equation

$$d_2(p,a) - k \cdot y(p) = d_2(q,a) - k \cdot y(q).$$

Without loss of generality, let S be contained in the y-positive halfplane of T. For each site $p \in S$, define the *circle of* p, $C(p)$, as the (Euclidean) circle with center p and radius $k \cdot y(p)$. Then by the equation above, $b(p,q)$ is just the set of points equidistant under the Euclidean distance from the two circles $C(p)$ and $C(q)$. In other words, the set of bisectors for S, under the skew distance, coincides with the set of bisectors for $\mathcal{C}(S) = \{C(p) \mid p \in S\}$, under the Euclidean distance. We hence conclude:

[a]Thanks go to Olivier Devillers for pointing out this property.

Theorem 1 *Let S be a set of point sites in the y-positive halfplane of T, and let $\mathcal{C}(S)$ be defined as above. Then $SV(S)$ is the Euclidean closest-point Voronoi diagram of $\mathcal{C}(S)$.*

It should be noted that $d(p,a)$ does *not* express the Euclidean distance $d_2(C(p),a)$ from $C(p)$ to a point a. For example, $d(p,a)$ may decrease when a is moving away from $C(p)$, which clearly cannot happen for $d_2(C(p),a)$.

Various properties are known for the Voronoi diagram of circles in the plane; see Section 4. In particular, the size of the diagram is linear, and there are efficient techniques for its construction. The $O(n \log n)$ time plane-sweep algorithm in Ref. [6] also is reasonably simple to implement. As $\mathcal{C}(S)$ can be computed from S in $O(n)$ time, we obtain:

Corollary 1 *For any fixed $k \geq 0$, the skew Voronoi diagram $SV(S)$ for a set S of n point sites can be computed in time $O(n \log n)$ and space $O(n)$.*

Since the number h of sites that have non-empty Voronoi regions in $SV(S)$ may be much smaller than n, the complexity given in Corollary 1 may not be optimal from the output-sensitive viewpoint. In Section 5, we shall present an optimal output-sensitive algorithm for computing $SV(S)$ in $O(n \log h)$ time and $O(n)$ space.

We mention here that another kind of direction-sensitive diagram, the so-called Voronoi diagram in a river, has been investigated in Ref. [15]. The underlying distance function is the time required to reach a point on the river by a boat starting from a site (and fighting against a constant river flow). Despite apparent similarity, the 'boat-sail' distance is different from the skew distance. For example, if the boat is faster than the river flow then the river Voronoi diagram is combinatorially equivalent to the Euclidean Voronoi diagram. If the river is faster than the boat then all the regions start at their defining sites. We will see quite different properties for skew Voronoi diagrams in the next section.

4. Geometry of Skew Voronoi Diagrams

Theorem 1 allows us to study the geometry of the skew Voronoi diagram $SV(S)$ by treating it as the Euclidean Voronoi diagram for a special set $\mathcal{C}(S)$ of circles obtained from S. We first briefly review some well-known facts on circle Voronoi diagrams. For proofs and further properties see Refs. [2,6,9,12,14].

Let $\mathcal{C}$ be any set of n circles on the plane. A circle $C \in \mathcal{C}$ is called *redundant* (in $\mathcal{C}$) if C is fully enclosed by some other circle in $\mathcal{C}$.

Property 1 *Let C and C' be non-redundant in $\mathcal{C}$. The bisector of C and C' is a branch of a hyperbola whose foci are the centers of C and C', and whose asymptotes are normal to the outer tangents of C and C'.*

By Property 1, for any fixed positive k the edges of $SV(S)$ are pieces of hyperbolas. The bisector $b(p,q)$ of two sites $p,q \in S$ degenerates to a vertical line if $y(p) = y(q)$. Note that $b(p,q)$ separates T into two portions which are star-shaped as seen from p and q, respectively. This implies that the regions of $SV(S)$ are star-shaped.

Let $V(\mathcal{C})$ denote the Euclidean Voronoi diagram of a set $\mathcal{C}$ of circles.

Property 2 *(1) The region of a circle $C \in \mathcal{C}$ in $V(\mathcal{C})$ is empty if and only if C is redundant in $\mathcal{C}$. (2) The region of C in $V(\mathcal{C})$ is unbounded if and only if C touches the boundary of the convex hull of $\mathcal{C}$.*

Recall from Theorem 1 that $\mathcal{C}(S)$ is a set of circles such that $V(\mathcal{C}(S)) = SV(S)$. Clearly, $\mathcal{C}(S)$ is not a general set of circles. In particular, it fulfills a property useful for characterizing unbounded regions of $SV(S)$. The easy proof is omitted.

Lemma 2 *Consider three sites $p, q, t \in S$ and some $k \geq 0$ such that none of the circles in $\{C(p), C(q), C(t)\}$ is redundant. Then t lies on the straight line through p and q if and only if $C(t)$ touches both of the two outer tangents of $C(p)$ and $C(q)$ (see Figure 3).*

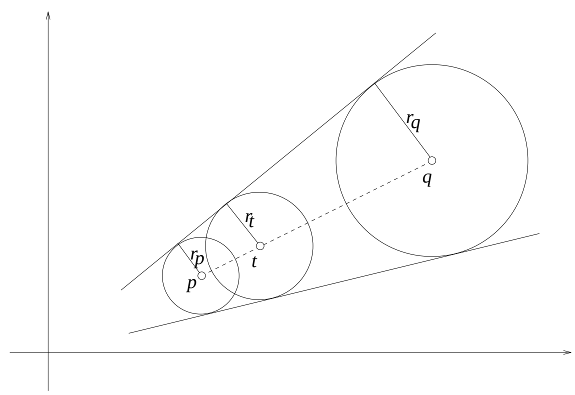

Fig. 3. Tangent property of $\mathcal{C}(S)$ ($k = \frac{1}{2}$).

In the rest of this section, we are mainly interested in the characterization of sites whose regions in $SV(S)$ are empty or unbounded. These questions have already been settled in Section 3 for the case $k < 1$. For simplicity, let us assume now that the x-coordinates $x(p)$ of the sites $p \in S$ are pairwise distinct.

4.1. Large Slope ($k > 1$)

The case of $k > 1$ is the most interesting one from the geometric point of view, due to anomalies like negative skew distance from a site p to a point a, and decreasing distance when a is moving away from p.

In Section 2, the negative area $N(p)$ of a site $p \in S$, and its boundary $L_0(p)$ have been defined. Let the *0-envelope*, $E_0(S)$, of S be the upper envelope of the graphs of all $L_0(p)$, when being seen as functions of the x-coordinate. Figure 1 shows a 0-envelope (as the dashed polygonal line) and the corresponding skew Voronoi diagram.

Lemma 3 *Let $p \in S$ and $k > 1$. Then $reg(p) \neq \emptyset$ if and only if $p \in E_0(S)$.*

Proof. First, let $p \in E_0(S)$. Then $d(q,p) \geq 0$ for all $q \in S \setminus \{p\}$. But $d(p,p) = 0$ such that $p \in reg(p)$.

Now assume $p \notin E_0(S)$, that is, p lies below $E_0(S)$. Then $p \in N(q)$ for some $q \in S$, and hence $d(q,p) < 0$. Consider an arbitrary point $a \in T$. By the triangle inequality, $d(q,a) \leq d(q,p) + d(p,a)$, so that $d(q,a) < d(p,a)$. Hence $a \notin reg(p)$, and we conclude that $reg(p) = \emptyset$. $\square$

Lemma 3 implies that the edges of $SV(S)$ lying above $E_0(S)$ have the structure of a forest (same for below $E_0(S)$): As $p \in reg(p)$ for each non-empty region, $E_0(S)$ cuts each such region into an upper and a lower part. Note also that any vertex of $E_0(S)$ which is not a site lies on the separator of the two neighboring sites on $E_0(S)$, and the separator has a vertical tangent at the vertex[b].

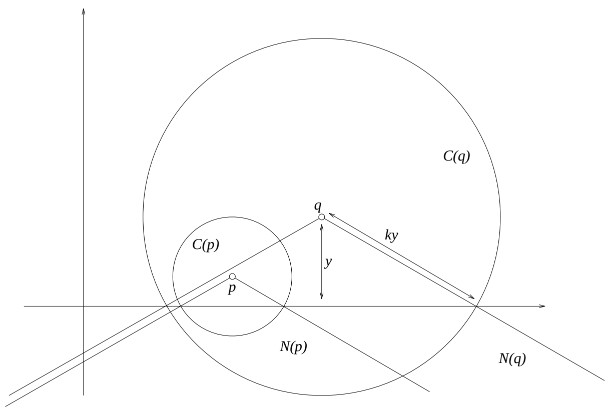

Fig. 4. Relationship between $N(p)$ and $C(p)$.

There is a geometric relationship between the negative area $N(p)$ and the circle $C(p)$ of a site $p \in S$. See Figure 4. As $k > 1$, $C(p)$ intersects the x-axis in two points u and v. Clearly, $d_2(p,u) = k \cdot y(p)$ and $d_y(p,u) = -y(p)$. (Same for v.) Hence $d(p,u) = d_2(p,u) + k \cdot d_y(p,u) = 0$ which means $u \in L_0(p)$. In other words, $C(p)$ intersects the x-axis in the same points as $L_0(p)$ does, and under an angle normal to the corresponding 0-rays that $L_0(p)$ is composed of. This fact can be used to show that $N(p) \subset N(q)$ if and only if $C(p) \subset C(q)$ which, by Property 2(1), gives another proof of Lemma 3. Recalling the slopes of 0-rays from Section 2, we see that all circles in $\mathcal{C}(S) = \{C(p) \mid p \in S\}$ intersect the x-axis under the same slopes $\pm\sqrt{k^2-1}$.

We now characterize sites with unbounded regions.

Lemma 4 *Let $p \in S$ and $k > 1$. $reg(p)$ is unbounded if and only if p lies on both $E_0(S)$ and the upper convex hull of S.*

Proof. We may assume $p \in E_0(S)$ as, by Lemma 3, $reg(p) = \emptyset$ otherwise, and nothing has to be shown.

To prove the 'if'-part, assume that p lies on the upper convex hull of S. Then there exists a straight line ℓ such that p lies above and $S \setminus \{p\}$ lies below ℓ. Moreover, by our assumption $p \in E_0(S)$, ℓ can be chosen to have an absolute slope less than that of $L_0(p)$. Now project S onto ℓ in y-direction and consider, for each site $q \in S$, the circle $C(q')$ of its projection q'. Due to the small slope of ℓ, Figure 4 tells us that these circles pairwise do not contain each other. In particular, they have a

[b]Thanks go to Günter Rote for pointing out these properties.

single common upper tangent τ, by Lemma 2. As the radius of $C(p')$ continuously increases when p' is moved upwards to p, $C(p)$ intersects the halfplane above τ. Similarly, for all other sites q, the radius of $C(q')$ continuously decreases when q' is moved downwards to q, such that $C(q)$ lies entirely below τ. This shows that $C(p)$ touches the boundary of the convex hull of $\mathcal{C}(S)$, and $reg(p)$ is unbounded then, by Property 2(2).

For the 'only if' part, suppose now that p does not lie on the upper hull. Then there are sites s, t on the upper hull such that p lies below the line segment $\overline{st}$. Because of $k > 1$, when $y(p)$ is decreasing, the radius of $C(p)$ shrinks faster than $y(p)$ itself. Lemma 2 now implies that $C(p)$ lies in the interior of the convex hull of $\{C(s), C(t)\}$, and thus of $\mathcal{C}(S)$. Hence $reg(p)$ is bounded by Property 2(2). $\square$

4.2. Moderate Slope ($k = 1$)

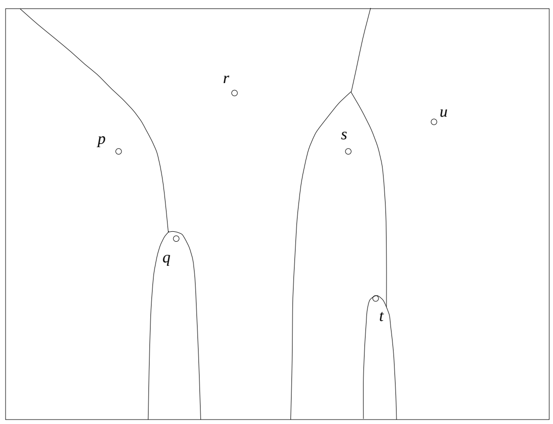

Fig. 5. Skew Voronoi diagram for $k = 1$.

The skew Voronoi diagram $SV(S)$ also has special properties in the case of $k = 1$. Once more, this can be seen easily from the associated set $\mathcal{C}(S)$ of circles. For each site $p \in S$, the radius of $C(p)$ now is just $y(p)$. That is, the x-axis is tangent to $C(p)$ for all $p \in S$. This implies that all circles touch the boundary of the convex hull of $\mathcal{C}(S)$. We obtain from Property 2(2):

Lemma 5 *For $k = 1$, all regions of $SV(S)$ are unbounded.*

The 0-rays of p degenerate to a single vertical ray emanating and extending below p. As $L_0(p) \subset reg(p)$, all regions of $SV(S)$ extend to infinity in negative y-direction; see Figure 5. This fact is also reflected by $\mathcal{C}(S)$. By Property 1, each region bisector has a vertical asymptote. So the left-to-right order of the regions reflects the x-sorted order of S. Similar to the case $k > 1$, $SV(S)$ has the structure of a forest.

5. Discussion and Extensions

We have introduced and investigated Voronoi diagrams under skew distances, and have shown that they have nice geometric and algorithmic properties. This section gives an output-sensitive algorithm for their construction, some generalizations of the concept, and a solution to the all nearest neighbors problem.

The following lemma enables us to compute skew Voronoi diagrams in an output-sensitive manner.

Lemma 6 *For any constant $k > 1$, the 0-envelope $E_0(S)$ of a set S of n sites can be computed in $O(n \log h)$ time and $O(n)$ space, where h is the number of sites on $E_0(S)$.*

Proof. We compute $E_0(S)$ by reducing the problem to that of finding the maximal dominating elements in a plane.[13] Let $L_{x'}$ and $L_{y'}$ be the two lines passing through the origin of T that are orthogonal to the 0-rays of the sites in S, respectively. Use $L_{x'}$ and $L_{y'}$ as the axes of a new coordinate system of T and recompute the coordinates of the sites of S accordingly. Then it is easy to see that the curve $E_0(S)$ can be specified by using the maximal dominating elements of S in this new system. By using the 'marriage-before-conquest' approach in Ref. [7], $E_0(S)$ can be computed in $O(n \log h)$ time and $O(n)$ space. $\square$

Combining Lemma 6 and Corollary 1 gives the following result which is optimal by reduction from the convex hull problem.

Corollary 2 *For any fixed $k \geq 0$, the skew Voronoi diagram for n sites can be computed in $O(n \log h)$ time and $O(n)$ space, where h is the number of non-empty Voronoi regions.*

Next we consider, for $k > 1$, a generalization of the skew Voronoi diagram problem which is defined as follows: Given a set S of n sites, let $S' = S$, compute $SV(S')$, and call it the *layer*-1 *diagram* of S, denoted by $SV_1(S)$; next, remove from S' all the sites that have non-empty regions in $SV_1(S)$ (that is, all sites on $E_0(S')$), and repeat the computation on S', which next gives $SV_2(S)$, and so on until $S' = \emptyset$.

All the 0-envelopes of the shrinking set S' can be computed beforehand, in overall time $O(n \log n)$ and space $O(n)$, by first performing the reduction in Lemma 6 and then applying the maxdominance algorithm in Ref. [1]. We then compute the skew Voronoi diagram separately for the sites on each such 0-envelope. In this way, all the Voronoi diagram layers of S are obtained in a total of $O(n \log n)$ time and $O(n)$ space.

$SV_1(S)$ and $SV_2(S)$ can be used to solve the all nearest neighbors problem under the skew distance: For each $p \in S$, find a site $q \neq p$ such that $d(q,p)$ is minimized. Note that q need *not* be any of the Voronoi neighbors of p in $SV(S)$. For example, when $E_0(S)$ contains only one site p of S, then p has no Voronoi neighbor in $SV(S)$ but still has a closest site $q \neq p$. We call q the *closest incoming neighbor* of p, denoted by *neigh-to*(p).

Our algorithm for the all nearest neighbors problem is based on the following simple observation: For any $p \in S$, *neigh-to*(p) is the site q such that $reg(q)$ in $SV(S \setminus \{p\})$ contains p. A naive algorithm based on this observation would take quadratic time. We will utilize the following lemma instead.

Lemma 7 *Let S_1 (resp., S_2) be the set of all sites in S that have non-empty regions in $SV_1(S)$ (resp., $SV_2(S)$). Then neigh-to(p) for every site $p \in S$ belongs to $S_1 \cup S_2$.*

Proof. Let $p \in S$. If p lies below $E_0(S)$ then $neigh\text{-}to(p) \in S_1$, by the definition of $SV_1(S)$ and Lemma 3, and hence the lemma is true. If p lies on $E_0(S)$ then $neigh\text{-}to(p)$ is on $E_0(S \setminus \{p\})$, by the observation stated before Lemma 7. But then the sites of S on $E_0(S \setminus \{p\})$ form a subset of $S_1 \cup S_2$, because p is on $E_0(S)$ and thus $p \in S_1$. $\square$

We point out that Lemma 7 does not rely on the specific structure of skew Voronoi diagrams; it requires only the general definition of layers of a Voronoi diagram. Hence our all nearest neighbors algorithm here can be applied to other types of Voronoi diagrams where sites with empty regions occur. The algorithm consists of the following steps: (1) Compute $SV_1(S)$ and $SV_2(S)$, (2) for each site p of S, if p is below $E_0(S)$, then do a point location in $SV_1(S)$[13] to obtain $neigh\text{-}to(p)$, (3) if p is on $E_0(S)$, then find the closest site q of p among those in $S_1 \setminus \{p\}$ (that is, from the Voronoi neighbors of p in $SV_1(S)$), find the site $q \in S_2$ whose Voronoi region in $SV_2(S)$ contains p (by a point location in $SV_2(S)$), and let $neigh\text{-}to(p)$ be the closer site from $\{q, q'\}$.

The correctness of the above algorithm follows from Lemma 7 and the observation stated before. Its complexity bounds are $\Theta(n \log(h_1 h_2))$ time and $O(n)$ space, where $h_1 = |S_1|$ and $h_2 = |S_2|$. Thus we get $\Theta(n \log n)$ in the worst case, and a possibly better runtime for special sets of sites.

The *farthest-site* skew Voronoi diagram of S can be defined to contain, for each site $p \in S$, the region at skew distance farthest from p. Similar to Theorem 1, this diagram coincides with the farthest-circle Euclidean Voronoi diagram of $\mathcal{C}(S)$. Again, the $O(n \log n)$ time and $O(n)$ space plane-sweep algorithm in Ref. [6] can be applied.

However, as shown in the lemma below, the farthest-site skew Voronoi diagram can also be computed in an output-sensitive manner. Let us consider the most interesting case $k > 1$ first. Let $L'_0(p)$ and $N'(p)$ be the reflection of $L_0(p)$ and $N(p)$, respectively, at site p and let $E'_0(S)$ be the *lower envelope* of $\{L'_0(p) \mid p \in S\}$. The sites on $E'_0(S)$ are just those having no sites in their negative areas.

Lemma 8 *Let $k > 1$. The region $reg_f(p)$ of $p \in S$ in the farthest-site skew Voronoi diagram of S is non-empty (and then unbounded) if and only if p lies on $E'_0(S)$ and on the lower convex hull of S.*

Proof. Assume first that p lies above $E'_0(S)$. Then there is a site q with $p \in N'(q)$. That is, $q \in N(p)$ and $d(p,q) < 0$. Consequently, for an arbitrary point a we have $d(p,a) \leq d(p,q) + d(q,a) < d(q,a)$. This shows $a \notin reg_f(p)$ and hence $reg_f(p) = \emptyset$.

Next, assume that p does not lie on the lower convex hull of S. Let p' denote the projection of p on the edge $\overline{st}$ of the lower hull. For an arbitrary point a, we know that $d(p', a) > d(p, a)$, since the circle $C(p')$ is totally contained in circle $C(p)$ provided $k > 1$. From $\max\{d(s,a), d(t,a)\} > d(p', a)$ we conclude $a \notin reg_f(p)$.

The proof of the 'if'-part follows the lines of the proof of Lemma 4. For a site p that lies on $E'_0(S)$ as well as on the lower convex hull of S, there exists a straight line ℓ such that (1) p lies below and $S \setminus \{p\}$ lies above ℓ, and (2) ℓ does not intersect $N(p)$. Projection of S onto ℓ in y-direction gives rise to circles that pairwise do not contain each other and that have a common upper tangent τ. Moving back projected sites to their original position shows that $C(p)$ lies entirely below τ, and that $C(q)$ intersects the halfplane above τ for all other sites q. That is, there are points sufficiently far in the direction normal to τ for which $C(p)$ is the farthest circle. This shows that $reg_f(p)$ is in fact non-empty and unbounded. $\square$

Note that, for $k = 1$, the proof above still goes through. Using the tangent argument, it is also easy to verify that, for $k < 1$, exactly the sites on the (entire) convex hull of S have non-empty and unbounded regions.

Generalizations of the skew Voronoi diagram based on the general L_p-*distance* are possible. The result in Theorem 1 still goes through: The induced diagrams are L_p Voronoi diagrams for L_p circles. These can be viewed as abstract Voronoi diagrams in the sense of Ref. [8], computable in $O(n \log n)$ time and $O(n)$ space. Also, Lemma 6 generalizes in a straightforward manner. 0-rays are still halflines, with slopes $\pm 1/\sqrt[p]{k^p - 1}$ for $p < \infty$ and $\pm 1/k$ for L_∞. Thus the complexity bounds in Corollary 2 hold for L_p-based skew Voronoi diagrams as well.

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