The Steiner Minimal Tree Problem in the λ-*geometry* Plane*

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Abstract

A Steiner Minimal Tree (SMT) for a given set P of points is a shortest network interconnecting the points of P whose vertex set may include some additional points in order to get the minimum possible total length in a metric space. When no additional points are allowed the minimum interconnection network is the well-known minimum spanning tree (MST) of P. The Steiner ratio is the greatest lower bound of the ratio of the length of an SMT over that of an MST of P. In this paper we study the Steiner minimal tree problem in which all the edges of SMT have fixed orientations. We call it the SMT problem in the λ-*geometry* plane, where λ is the number of possible orientations.

Here is the summary of our results.

1. We show that the Steiner ratio for $|P| \geq 3$ is $\frac{\sqrt{3}}{2}\cos(\pi/2\lambda)$, for $\lambda = 6m+3$ and integer $m \geq 0$, and is $\sqrt{3}/2$, for $\lambda = 6k$ and integer $k \geq 1$, disproving a a conjecture of Du *et al.*[3] that the ratio is $\sqrt{3}/2$ iff the unit disk in normed planes is an ellipse.
2. We derive the Steiner ratios for $|P| \leq 4$ for all possible λ's and show that for $|P| \geq 3$ there exists an SMT whose Steiner points lie in a *multi-level* Hanan-*grid*, generalizing a result that holds for rectilinear case, i.e., $\lambda = 2$.

These results show that the Steiner ratio is not a monotonically increasing function of λ, as believed by many researchers. We conjecture that the Steiner ratios obtained above ($|P| \leq 4$) are actually true for *all* $|P| \geq 3$.

1 Introduction

Let P be a set of n points in a metric space. A Steiner minimal tree (SMT) on P is a shortest network interconnecting P while allowing additional points, called *Steiner points*. Since the problem of finding such a tree has been shown to be NP-hard in Euclidean and rectilinear spaces[6, 7], polynomial time approximation algorithms were proposed. The *performance ratio* of any approximation $\mathcal{A}$ in metric space $\mathcal{M}$ is defined as

$$\rho_{\mathcal{M}}(\mathcal{A}) = \inf_{P \in \mathcal{M}} \frac{L_s(P)}{L_{\mathcal{A}}(P)}$$

where $L_s(P)$ and $L_{\mathcal{A}}(P)$ denote the lengths of a Steiner minimal tree and of the approximation $\mathcal{A}$ on P in space $\mathcal{M}$. One well-studied approximation interconnection network is the minimum spanning tree (MST) in which *no* Steiner points are allowed. When the approximation is the MST, the performance ration is known as the *Steiner ratio*, denoted simply as ρ. The following Steiner ratio results are known. In the L_2 norm $\rho = \sqrt{3}/2$[1], and in the L_1 norm, it is $2/3$[9]. In the L_2 norm, the edges connecting P are straight lines of artibrary orientation, while in the L_1 norm, they are restricted to be either horizontal or vertical. Let λ denote the number of possible orientations, and the unit disk D in this metric space, called the λ-geometry plane[12] is a centrally symmetric 2λ-gon. Fig. 1 shows the unit disk in various λ. It was shown in [12] that the Steiner ratio, denoted $\rho^\lambda(D)$, satisfies the inequality $\rho^\lambda(D) \geq \frac{\sqrt{3}}{2}\cos(\pi/2\lambda)$. Du *et al.* in [3] showed that in any normed plane where the unit disk is an arbitrary compact convex centrally symmetric domain $\rho(D)$ satisfies the inequality $0.623 < \rho(D) < 0.8686$. They conjectured[3]

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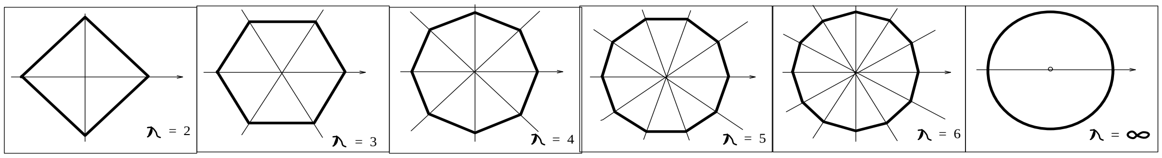

Fig. 1. Examples of unit disks in the λ-*geometry* plane.

that $2/3 \leq \rho(D) \leq \sqrt{3}/2$ for any norm and that $\rho(D) = \sqrt{3}/2$ if and only if D is an ellipse and $\rho(D) = 2/3$ if and only if D is a parallelogram. It was shown in [3] that in any normed plane $2/3 \leq \rho(D)$ for $|P| \leq 6$, and in [11] that $\rho(D) \leq \sqrt{3}/2$ for any L_p norms.

We show in this paper that $\rho^\lambda(D) = \frac{\sqrt{3}}{2}\cos(\pi/2\lambda)$, for $\lambda = 6m+3$ and integers $m \geq 0$. That is, the lower bound derived in [12] is tight for those λ's. When $\lambda = 6k$, $\rho^\lambda(D) = \sqrt{3}/2$, disproving the conjecture that D must be an ellipse. We also derive the ratios $\rho^\lambda(D)$ for $|P| \leq 4$. These results show that the Steiner ratio is not a monotonically increasing function of λ, as believed by many researchers.

In the L_1 norm ($\lambda = 2$), it is well-known that the Steiner points of any Steiner minimal tree must lie on a grid, known as the *Hanan*-grid, defined by drawing horizontal and vertical lines through the given points. Sarrafzadeh and Wong[12] showed that in the λ-geometry plane, the *Hanan*-grid does not necessarily contain all possible Steiner points in any Steiner minimal tree, but did not offer any characterization of where the Steiner points should lie. We show that their exists a Steiner minimal tree in which the Steiner points can be chosen from a *multi-level* Hanan-grid defined by P, generalizing the Hanan-grid result (which holds for $\lambda = 2$) to all λ's $< \infty$. Thus this result implies that there is a finite solution for the Steiner minimal tree problem in the λ-geometry plane. In [10] we have reported a similar result for $\lambda = 4$.

The paper is organized as follows. In Section 2 we give the basic definition and terminology along with some preliminary results. Section 3 gives derivations of ratios for smaller point sets. In Section 4 we prove our main result on Steiner ratios and in Section 5 we describe our multi-level grid theorem. Finally we make a conclusion.

2 Preliminaries

The given points in P are called *regular points*. The additional points which may exist in an SMT are called *Steiner points*. In the λ-*geometry* plane, each edge in an SMT connecting two points is allowed to make an angle $(i * \pi/\lambda)$ with the positive x-axis for integers $0 \leq i < \lambda$. The orientation of angle $(i * \pi/\lambda)$ with the positive x-axis is referred to as the i^{th} λ-direction, and the i^{th} and $(i+1st)$ λ-directions are said to be *adjacent*.

The Steiner ratio for a set P of n points in the λ-*geometry* plane $\mathcal{E}^\lambda$ is defined as

$$\rho_n^\lambda = \inf_{P \in \mathcal{E}} \frac{L_{smt}(P)}{L_{mst}(P)}.$$

where $L_{smt}(P)$ and $L_{mst}(P)$ are the lengths of an SMT and MST on P respectively.

From here on it is understood that the point sets are in $\mathcal{E}^\lambda$ and the specification will be omitted. For any two points p, q, if the straight line segment $\overline{p,q}$ connecting two points p and q has the i^{th} λ-direction for some i, then these two points or the edge connecting them are said to be *straight*. Otherwise they are said to be *nonstraight*. If p and q are nonstraight, then the edge connecting them can be realized by a *canonical path* consisting of exactly *two* segments with orientations equal to two adjacent λ-directions. There are two ways to connect two nonstraight points, and these two nonstraight edges form a parallelogram. One nonstraight edge can be obtained from the other by a *flipping* operation. We use $e(p, q)$ to denote an edge, straight or nonstraight. The distance between p and q is measured by the length of $e(p, q)$.

Let G_n denote a Steiner minimal tree of a set of n regular points. The following are fundamental properties of G_n which are generalizations of that of rectilinear SMT[5, 8, 12].

Property 1 *The number of Steiner points in G_n is at most $n-2$.*

Property 2 *The degree of any Steiner point q satisfies $3 \leq deg(q) \leq 4$. When $deg(q) = 4$, these four adjacent points form a cross.*

Property 3 *There is no acute angle nor straight angle formed by any two edges in G_n. Moreover there exists a G_n such that the three angles, $\alpha_1, \alpha_2, \alpha_3$, around a degree-3 Steiner point are:*
$\alpha_1 = \alpha_2 = \alpha_3 = 2\pi/3$, *for* 2λ *MOD* $3 = 0$;
$\alpha_1 = \alpha_2 = \lfloor 2\lambda/3 \rfloor(\pi/\lambda)$ *and* $\alpha_3 = \lceil 2\lambda/3 \rceil(\pi/\lambda)$, *for* 2λ *MOD* $3 = 1$; *or*
$\alpha_1 = \lfloor 2\lambda/3 \rfloor(\pi/\lambda)$ *and* $\alpha_2 = \alpha_3 = \lceil 2\lambda/3 \rceil(\pi/\lambda)$, *for* 2λ *MOD* $3 = 2$.
This is referred to as the **even angle property.**

We call the edge connecting two Steiner points, a *Steiner edge*, and that connecting one Steiner point and a regular point a *regular edge*. The *link distance* between two regular points is the number of Steiner edges on the path in G_n connecting them. A hexagonal tree $\mathcal{H}_n$ of n regular points is similar to a full Steiner tree except that all the segments in $\mathcal{H}_n$ have exactly *three* orientations, not necessarily those of λ-directions, and exactly *three* edges meet at each *junction* such that the angles between adjacent segments incident to a junction have at most one acute angle. Consider three nonparallel lines such that they intersect pairwise at vertices A, B and C. Through vertices A, B and C we draw lines parallel to segments $\overline{B,C}$, $\overline{C,A}$, and $\overline{A,B}$ respectively. The structure formed by repeating this operation is called a *Δ-lattice* with a base triangle $\Delta(A, B, C)$, and denoted $\mathcal{L}_\Delta$. The vertices of $\mathcal{L}_\Delta$ are called *lattice points*. The lattice whose base triangle is an equilateral triangle of unit length is called a *unit lattice*. The following results were proved by Du *et al.*[3].

Theorem 1. *Let ρ_0 be a positive number. Then for any norm with unit disk D, $\rho(D) \geq \rho_0$ if and only if for all hexagonal norm with unit disk $\mathcal{H}$, $\rho(\mathcal{H}) \geq \rho_0$.*

Lemma 2. *Let C be a critical set[a] of n lattice points of a unit lattice such that its MST is of length $n-1$ and let $\mathcal{H}(C)$ be a minimum hexagonal tree on C. $\mathcal{H}(C)$ can be transformed into an MST of C by a sequence of translating and flipping operations such that the length of $\mathcal{H}(C)$ is no less than the length of MST of C.*

When G_n has exactly $n-2$ Steiner points, it is a *full* Steiner tree, denoted G_n^*. G_n^* must have one of the local structures shown in Fig. 2(a), when $n > 4$. The two regular points that are connected to a common Steiner point are said to form a *regular pair*. It is obvious that $G_n^*, n \geq 4$, must have at least two regular pairs. The upper left diagram in Fig. 2(a) shows two regular pairs with a link distance exactly 2; and the other shows the two regular pairs with a link distance at least 2.

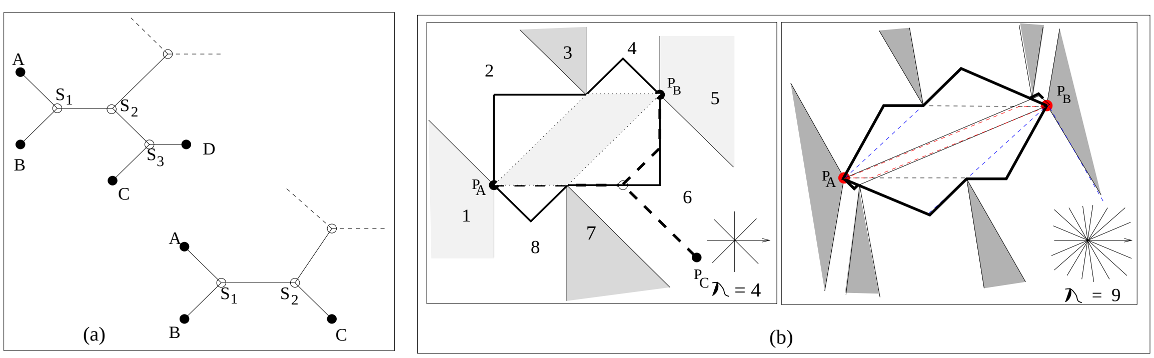

Fig. 2. (a) Two local structures in full SMTs. (b)Steiner loci of two regular points.

[a] C must satisfy the condition that there exists an MST that divides the interior of the n-gon formed by connecting adjacent lattice points, into $n-2$ equilateral triangles.

3 Steiner Ratios for Sets of Small Size

3.1 Steiner Ratios for λ-Equilateral Triangles

We show the Steiner ratio for three points that form a λ-equilateral triangle. Note that G_3 may *not* be unique. According to Property 3, a G_n that satisfies the even angle property can always be found and thus is used to prove the Steiner ratio.

Given a G_3 for points A, B, C, angle $\angle AqB$ is said to be *opposite* to edge $e(q, C)$. We shall use $\lambda = 4$, the 45°-SMT as an example for illustrations only. Similar results can be obtained for other $\lambda > 2$ and the proofs can be carried over with some modifications.

Lemma 3. *In G_3 at most one nonstraight edge, say edge $e(B, q)$, is incident on q. Furthermore the angle opposite to the nonstraight edge equals* 135° ($\lceil 2\lambda/3 \rceil(\pi/\lambda)$) (Fig. 2(b)). *In addition, the segment of the nonstraight edge not incident to q (called the second segment) is not parallel to $e(q, A)$ or $e(q, C)$.*

Lemma 3 is proved by a constructive method. For points A, B, we construct two rectangles $R_0(A, B)$ and $R_{+1}(A, B)$ as shown in Fig. 2(b). The boundary of the union of the two rectangles $\mathcal{L}_{A,B} = \partial(R_0(A, B) \bigcup R_{+1}(A, B))$ is the *Steiner locus* of A, B.

When point C lies outside of the region defined by $\mathcal{L}_{A,B}$, the Steiner point must lie on $\mathcal{L}_{A,B}$. If C lies inside, C itself is the Steiner point. From $\mathcal{L}_{A,B}$ we partition the plane into 8 regions as shown in Fig. 2(b), each of which is associated with a boundary segment of $\mathcal{L}_{A,B}$ that contains q. Note that for other values of λ the Steiner locus is defined by $(\lambda - \lfloor 2\lambda/3 \rfloor)$ parallelograms whose intersection is the parallelogram defined by the two canonical paths connecting p_A and p_B.

When $\varDelta(A, B, C)$ is an equilateral triangle, $\rho_3^\lambda = \frac{1}{4-2\sqrt{2}}$ for $\lambda = 4$. To see this we assume the Steiner point coincides with the origin O and $e(O, A)$ coincides with the x-axis, and edges $e(O, B), e(O, C)$, are symmetric to the x-axis, such that they form angles satisfying the angle property. For an arbitrary 3-point set it can be shown that the ratio of its L_{smt} and L_{mst} is always greater than that of a λ-equilateral triangle.

Lemma 4. *If λ is even, edge $e(B, C)$ is straight. Otherwise, it is nonstraight and the two segments $e(B, k)$ and $e(C, k)$ are of equal length, and form an angle $\pi - \frac{\pi}{\lambda}$* (Fig. 4).

We give below the formulae for the ratios.

<u>[1] When $\lambda = \{6m | m \geq 1, m \in I\}$:</u>

$$\rho_3^\lambda = \frac{\sqrt{3}}{2} \qquad \text{(same as the Euclidean Steiner ratio)}$$

<u>[2] When $\lambda = \{6m + 1 | m \geq 0, m \in I\}$:</u>

$$\rho_3^\lambda = \frac{3\cos(\pi/2\lambda) + \frac{\sin(\pi/\lambda)[\cos(\pi/2\lambda)\sin(\gamma) - 2\sin(\beta)\sin(\alpha)]}{\sin(\beta)(\sin(\beta) - \sin(5\beta))}}{4\sin(\alpha)}, \text{ where } \alpha = 2m\pi/\lambda,\ \beta = m\pi/\lambda,\ \gamma = (4m+1)\frac{\pi}{\lambda}$$

In particular, $\rho_3^7 \simeq 0.85824$

<u>[3] When $\lambda = \{6m + 2 | m \geq 0, m \in I\}$:</u>

$$\rho_3^\lambda = \frac{\frac{[2\sin(\pi/\lambda)\cos(\beta) + \sin(\beta) - \sin(\gamma)]\sin(\gamma)}{\sin(\pi/\lambda)\cos(\beta) + \sin(\beta)[\sin(\beta) - \sin(\gamma)]} + 2}{4\sin(\alpha)}, \text{ where } \alpha = (2m+1)\frac{\pi}{\lambda},\ \beta = m\pi/\lambda,\ \gamma = (m+1)\frac{\pi}{\lambda}$$

In particular, $\rho_3^2 = 3/4$ and $\rho_3^8 \simeq 0.8603882638$

<u>[4] When $\lambda = \{6m + 3 | m \geq 0, m \in I\}$:</u>

$$\rho_3^\lambda = \frac{\sqrt{3}}{2}\cos\frac{\pi}{2\lambda}$$

<u>[5] When $\lambda = \{6m + 4 | m \geq 0, m \in I\}$:</u>

$$\rho_3^\lambda = \frac{\frac{\sin(\beta)}{\sin(\gamma)} + \frac{[2\sin(\alpha)\sin(\gamma) - \cos(\gamma)]\sin(\pi/\lambda)}{[\sin(\gamma) - \sin(\beta)]\sin(\gamma)} + 2}{4\sin(\alpha)}, \text{ where } \alpha = (2m+1)\frac{\pi}{\lambda},\ \beta = m\pi/\lambda,\ \gamma = (m+1)\frac{\pi}{\lambda}$$

In particular, $\rho_3^4 = \frac{2+\sqrt{2}}{4} \simeq 0.85355339$ and $\rho_3^{10} \simeq 0.86038826$

[6] When $\lambda = \{6m+5 | m \geq 0, m \in I\}$:

$$\rho_3^\lambda = \frac{3\cos(\pi/2\lambda) + \frac{2\sin(\pi/\lambda)[\cos(\pi/2\lambda)\cos(\gamma) - \sin(\alpha)]}{\sin(\beta)-\sin(\gamma)}}{4\sin(\alpha)}, \text{ where } \alpha = (2m+2)\frac{\pi}{\lambda},\ \beta = m\frac{\pi}{\lambda},\ \gamma = (m+1)\frac{\pi}{\lambda}$$

In particular, $\rho_3^5 = \frac{5-2\cos(\pi/5)}{4} \simeq 0.8454$ and $\rho_3^{11} \simeq 0.8621377$

3.2 Steiner Ratios for $n = 4$

It suffices to consider the set P of lattice points in $\mathcal{L}_\Delta$, where Δ is a λ-equilateral triangle (cf. Theorem 1). See Fig. 3 for an illustration for $\lambda = 4$. Basically there are two possible critical sets, both defining a parallelogram. When $\lambda = 2m$, there are two topologically different solutions. One has *one* Steiner point

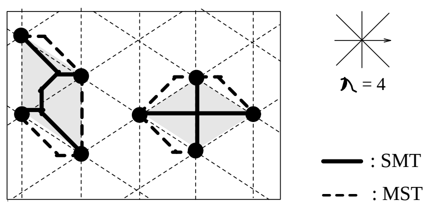

Fig. 3. An example of a unit-lattice for λ= 4, and two SMT's

of degree 4 (forming a *cross*) and the other has *two* Steiner points, denoted $G_4^{\lambda+}$ and $G_4^{\lambda*}$ respectively. The computation of the SMT's is based on the fact that the Steiner points must be determined by the Steiner loci of two regular pairs and its derivation is purely mechanical. These values all confirm the fact that their ratios are all greater than those for λ-equilateral triangles. Let $\rho_4^{\lambda+}$ and $\rho_4^{\lambda*}$ denote the Steiner ratios of $G_4^{\lambda+}$ and $G_4^{\lambda*}$ respectively. Below is a summary of the relationship among ρ_3^λ, $\rho_4^{\lambda+}$ and $\rho_4^{\lambda*}$. Except for $\lambda = 2$ (whose ratio is known to be $2/3$, and the SMT is a cross), ρ_3^λ is always a lower bound. In particular, $\rho_4^{\lambda+} = \rho_4^{\lambda*} = \frac{4-\sqrt{2}}{3} > \frac{1}{4-2\sqrt{2}} = \rho_3^\lambda$.

$\lambda = 2,$	$\rho_4^{\lambda+} < \rho_3^\lambda < \rho_4^{\lambda*}$
$\lambda = (6m+2)$	$\rho_3^\lambda < \rho_4^{\lambda*} < \rho_4^{\lambda+}$
$\lambda = 3,$	$\rho_3^\lambda < \rho_4^\lambda$
$\lambda = (6m+3)$	$\rho_3^\lambda < \rho_4^{\lambda*}$
$\lambda = 4,$	$\rho_3^\lambda < \rho_4^{\lambda*} = \rho_4^{\lambda+}$
$\lambda = (6m+4)$	$\rho_3^\lambda < \rho_4^{\lambda*} < \rho_4^{\lambda+}$
$\lambda = 5, (6m+5)$	$\rho_3^\lambda < \rho_4^{\lambda*}$
$\lambda = 6, (6m)$	$\rho_3^\lambda < \rho_4^{\lambda*}$
$\lambda = 7, (6m+1)$	$\rho_3^\lambda < \rho_4^{\lambda*}$

4 Proof of Steiner Ratios when λ is Multiple of 3

We briefly review the proof due to Du and Hwang[1, 4] of the Gilbert-Pollak conjecture that the Euclidean Steiner ratio $\rho^\infty = \frac{\sqrt{3}}{2}$.

They first transform the problem into a minimax problem so that one needs only to verify the ratio for a critical set C (Lemma 2) of points drawn from a unit Δ-lattice $\mathcal{L}_\Delta$. They then convert an SMT into a hexagonal tree $\mathcal{H}(C)$. The three orientations of $\mathcal{H}$ are parallel to those of Δ. In doing so, they showed that the length of SMT satisfies:

$$L_{smt}(C) \geq \frac{\sqrt{3}}{2} * L_{\mathcal{H}}(C),$$

where $L_{\mathcal{H}}(C)$ denotes the length of $\mathcal{H}(C)$. $\mathcal{H}(C)$ is then transformed so that the junction points of $\mathcal{H}(C)$ coincide with the lattice points, thus producing an MST. In this process they showed that $L_{\mathcal{H}}(C)$ is no less than $L_{mst}(C)$. Therefore

$$L_{smt}(C) \geq \frac{\sqrt{3}}{2} * L_{\mathcal{H}}(C) \geq \frac{\sqrt{3}}{2} * L_{mst}(C).$$

$$\rho = \frac{L_{smt}}{L_{mst}} \geq \frac{\sqrt{3}}{2}.$$

We follow the same strategy in deriving the Steiner ratio ρ^{λ} for $\lambda = 3m$, and integer $m \geq 1$. To show a lower bound on the Steiner ratio it suffices from Theorem 1 to show that the lower bound holds for a critical set C of n points drawn from a unit lattice. Let L_{Δ} denote the unit lattice in the Euclidean sense. Δ, shown in dotted lines, is the same as the actual λ-equilateral triangle, except for cases when $\lambda = 6m + 3$ for some integer $m \geq 0$ (Fig. 4(a)). Since for these λ's the Steiner angles in G_n^* are all 120°, it is itself a hexagonal tree. The three orientations of this tree may not be the same as those of Δ, we convert each Steiner edge $e(p, q)$ by at most two segments, $e(p, x), e(x, q)$, parallel to two sides of Δ (see Fig. 4(b)).

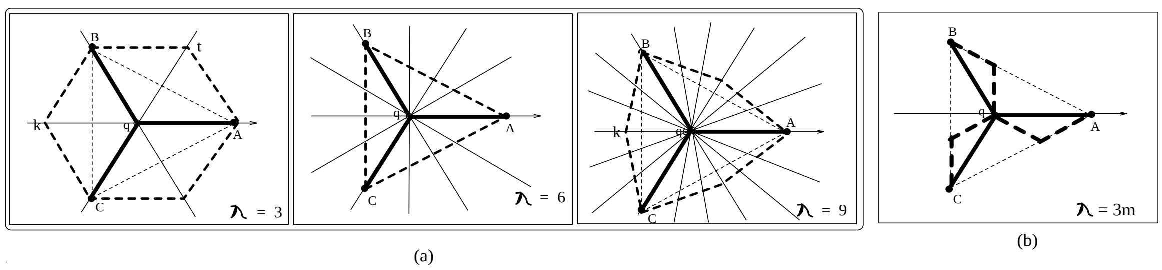

Fig. 4. An example of lattice and hexagonal trees in the λ-*geometry* plane, where $\lambda = 3m$.

Since $\angle pxq = 120°$, we have

$$L_{e(p,q)} \geq \frac{\sqrt{3}}{2} * (L_{e(p,x)} + L_{e(q,x)})$$

$$L_{smt}(C) = \Sigma\{L_{e(p,q)} | e(p,q) \in G_n^*\} \geq \frac{\sqrt{3}}{2} * \Sigma\{(L_{e(p,x)} + L_{e(q,x)})\}$$

After that we obtain a hexagonal tree which can be transformed into another hexagonal tree $\mathcal{H}'(C)$, whose junctions coincide with the lattice points, i.e., an MST for C, of length no greater than that of the hexagonal tree.

$$L_{smt}(C) \geq \frac{\sqrt{3}}{2} * L_{\mathcal{H}'}(C)$$

For $\lambda = 6m$ the length of $\mathcal{H}'(C)$ is the same as $L_{mst}(C)$, we have $\rho^{\lambda} \geq \frac{\sqrt{3}}{2}$. For $\lambda = 6m + 3$ for some integer $m \geq 0$, $L_{\mathcal{H}'}(C) = L_{mst}(C) * \cos\frac{\pi}{2\lambda}$. Thus we have $\rho^{\lambda} \geq \frac{\sqrt{3}}{2} * \cos\frac{\pi}{2\lambda}$, and for $\lambda = 3$ we have $\rho^{\lambda} \geq \frac{3}{4}$. Since the bound is realized by three points, we conclude

Theorem 5. *In the λ-geometry plane, the Steiner ratio is $\rho^{\lambda} = \frac{\sqrt{3}}{2} * \cos\frac{\pi}{2\lambda}$ for $\lambda = 6m + 3$ for some integer $m \geq 0$, and $\rho^{\lambda} = \frac{\sqrt{3}}{2}$ for $\lambda = 6m$.*

5 Multi-Level Grid Theorem

Given a set P of regular points, the Hanan-grid is the structure obtained by drawing horizontal and vertical lines through each regular point and the intersections of these lines are called *grid* points. It is known that the Steiner points for an SMT of P, for $\lambda = 2$, must occur at the the grid points. Sarrafzadeh and Wong[12] showed that in the λ-geometry plane, the Hanan-grid may not contain all possible candidates of Steiner points when $\lambda > 2$. In fact, Du and Hwang[2] conjectured that the Hanan-grid may not contain all possible candidates of Steiner points if the unit-disk in d-dimemsion space has more than $2d$ extreme points. However, as we outline below (Theorem 6) that for $\lambda = 4$ the Steiner points can be located in a "multi-level grid" so that the number of candidate Steiner points for G_n is finite. This is true for any $\lambda \neq \infty$.

For $\lambda = 4$ we can construct a *multi-level grid* as follows. (For other values of λ we do it similarly.) For each regular point p draw four straight lines of slopes $0, +1, -1$ and ∞. These $4n$ lines form the *1st-level* grid. In general, the next-*level* grid is formed by repeating the same procedure at each of the *new* grid points generated at the previous *level*. Let $\mathcal{G}_k$ denote the set of grid points at levels up to k, where k is an integer.

Theorem 6. *There exists a G_n such that the Steiner points are grid points in $\mathcal{G}_{n-2}$, for $n \geq 3$.*

Theorem 6 is referred to as the *multi-level* grid theorem. From now on, we consider only a full Steiner tree because a non-full Steiner tree can be separated into several full Steiner sub-trees.

Recall from Section 3.1 that for each pair of regular points we can construct its Steiner locus. The Steiner point for a set consisting of this pair and the third point must be on the boundary of the locus. From Lemma 3 among the three regular edges incident to the Steiner point at most one is nonstraight. In other words the Steiner point lies on the 1st level grid determined by the two regular points with straight regular edges. We shall now use this result to show that the theorem is true for point sets of size $n = 4$, and then prove by induction (omitted) that it is true for all n.

Lemma 7. *There exists a G_4^*, such that one of the two Steiner points can be located on the 1st-level grid.*

Proof. Consider the full topology of a G_4^* that has two Steiner points. Note that the number of possible topologies of G_4 is 5. The relationship between each topology and its loci intersection is shown in Figure 5.

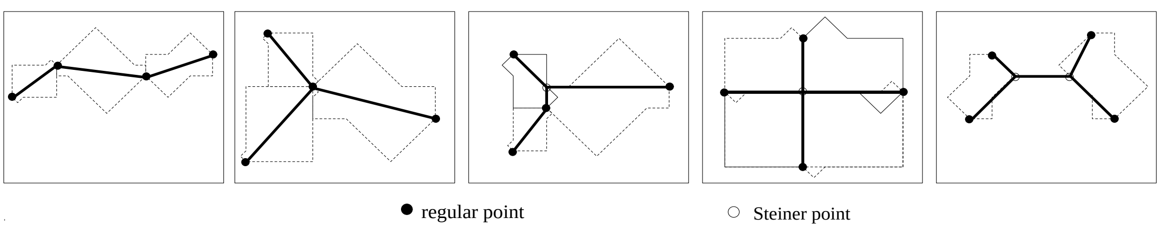

Fig. 5. Five possible topologies (solid lines) of G_4 and the relationships of their loci (dotted lines) intersection.

Consider G_4^* defined by two regular pairs $[A, B]$ and $[C, D]$ connected to Steiner points s_1 and s_2 respectively. (Fig. 6). Suppose that the Lemma is false. That is, each of Steiner points (s_1 and s_2) has one nonstraight regular edge and the Steiner edge must be straight (cf Lemma 3). It is sufficient to consider the case when the two Steiner loci, $\mathcal{L}_{A,B}$ and $\mathcal{L}_{C,D}$, are disjoint. It is obvious that s_1 cannot be inside $\mathcal{L}_{C,D}$ and s_2 cannot be inside $\mathcal{L}_{A,B}$. Let us position $e(s_1, s_2)$ on the x-axis. There are three possible configurations for the location of s_1 and s_2 on the Steiner loci $\mathcal{L}_{A,B}$ and $\mathcal{L}_{C,D}$. Assume that s_1 lies on the boundary segment of $\mathcal{L}_{A,B}$ of slope $+1$. Then the possible slopes of the boundary segment of $\mathcal{L}_{C,D}$ on which s_2 can lie are $+1$, -1 and ∞, shown respectively in Fig. 6(a), (b), and (c). Define a *parallel sliding* operation of the Steiner edge $e(s_1, s_2)$ by shifting $e(s_1, s_2)$ parallelly by a distance ϵ while maintaining the same orientations of the segments incident on the Steiner points and that s_1 and s_2 lie on $\mathcal{L}_{A,B}$ and $\mathcal{L}_{C,D}$ respectively. We say that these two Steiner trees are *topologically equivalent*. It can be easily verified these two trees in Fig. 6(a) are of the same length. They are said to be *topologically congruent*, while the configurations in Fig. 6(b) and (c) are of different length. Notice that in case (a), if the shifting direction is to reduce the length of the second segment of regular edge $e(A, s_1)$, for instance, by ϵ, the length of the second segment of regular edge on the opposite side (edge $e(C, s_2)$) is increased by the same amount.

Consider the following two cases: (1) The regular points of the two nonstraight edges are either both above the x-axis or both below. (2) The regular points of the two nonstraight edges lie on different sides of the x-axis.

We can show that in (1) the SMT can always be shortened which is a contradiction (Fig. 6(b)) and that in (2) one of the Steiner points, e.g. s_1 (or s_2), can be moved to a vertex, a $1st$-level grid point, of its locus $\mathcal{L}_{A,B}$ (or $\mathcal{L}_{C,D}$).

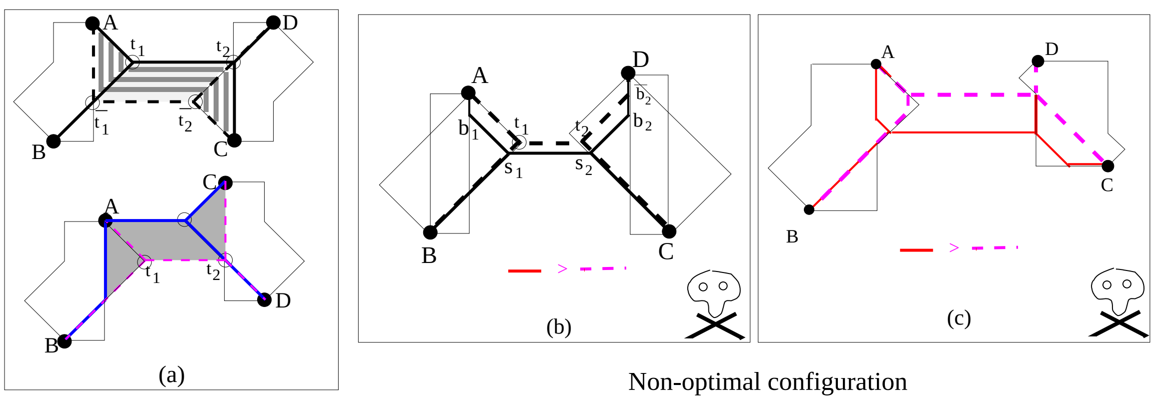

Fig. 6. Three possible configurations and the parallel sliding operation.

That is, in G_4^*, if its solution is unique, both Steiner points are $1st$-level grid points. Otherwise, there are infinitely many topologically congruent G_4^*'s, and there exist two extreme solutions which contain at least one Steiner point at a $1st$-level grid point.

Lemma 8. *Suppose the two closest regular pairs in $G_n^*, n \geq 4$ are connected by a Steiner path $s_1, ..., s_{k+1}$, $k \geq 1$. Assume that none of the Steiner points on this Steiner path lies on the 1st-level grid. That is, each regular pair has a nonstraight regular edge, and the other regular edges are all straight. We can transform a nonstraight edge by length ϵ from one end of the Steiner path to the other without changing the total length of SMT, if and only if k is even (resp. odd) and the two nonstraight edges are on the same (resp. different) side of the Steiner path.*

The proof of this lemma can be shown by using parallel sliding operations. If the lemma were false, the sliding operation will either make a regular point coincide with a Steiner point or have the total length reduced. We omit its details.

6 Remarks and Conclusion

We have shown that the Steiner ratio for a set of $n \geq 3$ points in the λ-geometry plane is $\frac{\sqrt{3}}{2}\cos(\pi/2\lambda)$, for $\lambda = 6m + 3$ and integer $m \geq 0$, and is $\sqrt{3}/2$, for $\lambda = 6k$ and integer $k \geq 1$, disproving a a conjecture made by Du *et al.*[3]. We also have shown that there exists a Steiner minimal tree whose Steiner points must lie on a multi-level Hanan-grid. Our initial study of the Steiner ratios for point sets of size 3 and 4 leads us to conjecture that in the λ-geometry plane, the Steiner ratio is achieved by 3 points that form a λ-equilateral triangle. We will focus our future research on settling this conjecture.

Fig. 7 and Table 1 summarize the λ-*geometry* Steiner ratio results along with the lower bound result of Sarrafzadeh and Wong[12].

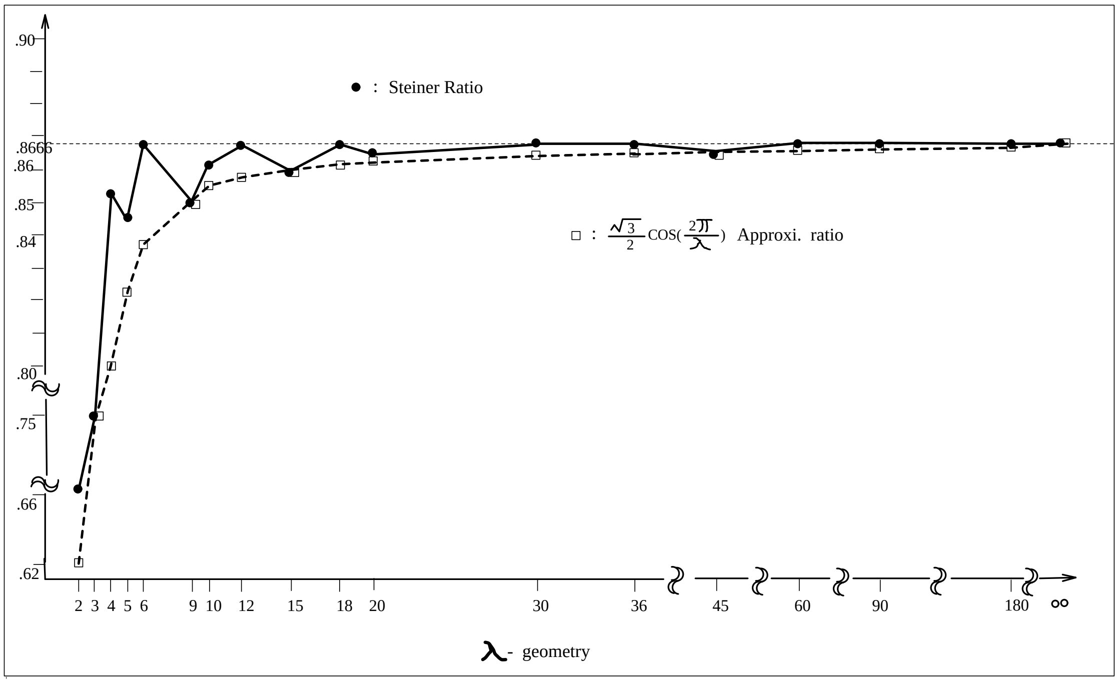

Fig. 7. Steiner Ratio and an approximation function in the λ-*geometry* plane.

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λ	2	3	4	5	6	9	10	12	15
$\theta°$	90°	60°	45°	36°	30°	20°	18°	15°	12°
2λ MOD 3	1	0	2	1	0	0	2	0	0
Equil-△ or Equil-◇	$\frac{3}{4}$ $\frac{2}{3}$	$\frac{3}{4}$	$\frac{2+\sqrt{2}}{4}$	$\frac{5-2\cos(\pi/5)}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}\cos(\pi/18)}{2}$	0.86327	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}\cos(\pi/30)}{2}$
$\frac{\sqrt{3}}{2}\cos(\frac{\pi}{2\lambda})$ $\simeq$	0.623	0.75	0.80	0.823	0.836	0.852	0.855	0.858	0.861
Steiner Ratio $\simeq$	0.666	0.75	0.8535	0.8454	0.8666	0.852	0.86327	0.8666	0.861

λ	18	20	30	36	45	60	90	180	∞
$\theta°$	10°	9°	6°	5°	4°	3°	2°	1°	Euclidean
2λ MOD 3	0	0	1	0	0	0	0	0	0
Equil-△ or Equil-◇	$\frac{\sqrt{3}}{2}$	0.86518	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}\cos(\pi/90)}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\sqrt{3}}{2}\cos(\frac{\pi}{2\lambda})$ $\simeq$	0.862	0.863	0.864	0.8651	0.8654	0.8657	0.8658	0.8659	0.8666
Steiner Ratio $\simeq$	0.8666	0.86518	0.8666	0.8666	0.8657	0.8666	0.8666	0.8666	0.8666

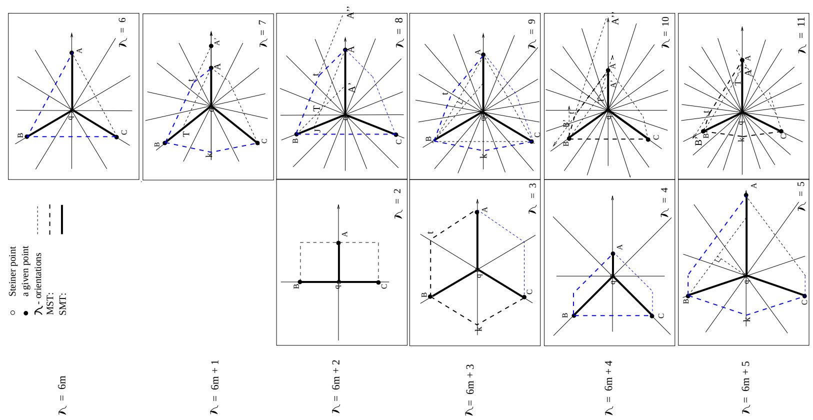

Table 1: Steiner Ratio and examples of equilateral triangles in the λ-*geometry* plane.