Alpha-Beta Pruning: Algorithm and Analysis

Tsan-sheng Hsu

徐讚昇

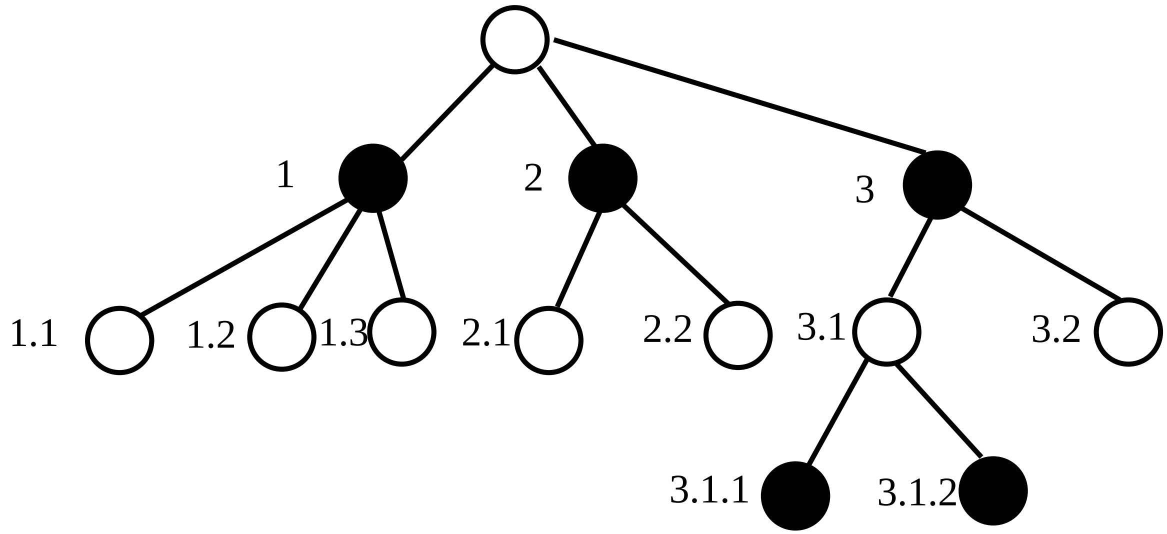
tshsu@iis.sinica.edu.tw

http://www.iis.sinica.edu.tw/~tshsu

Introduction

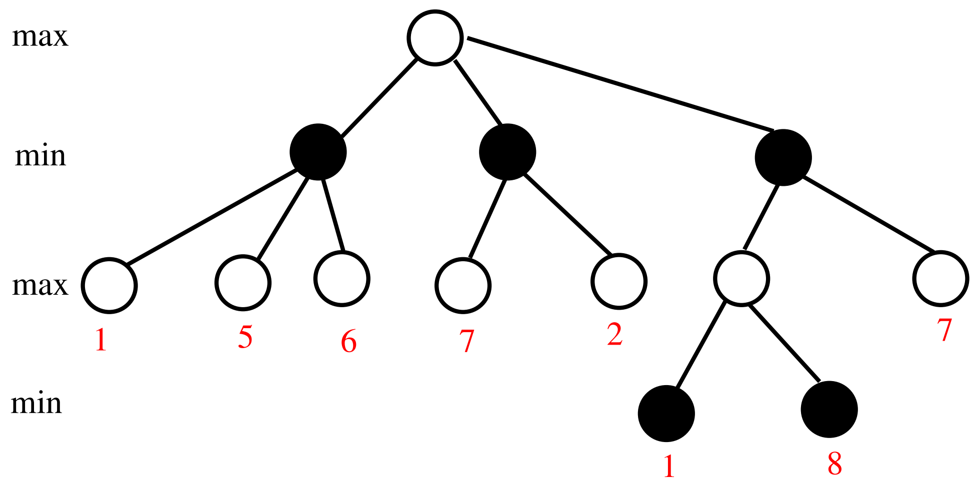
- **Alpha-beta pruning is the standard searching procedure used for solving 2-person perfect-information zero sum games exactly.**
- **Definitions:**
  - **A *position* p.**
  - **The value of a position p, $f(p)$, is a numerical value computed from evaluating p.**
    - ▷ ***Value is computed from the root player's point of view.***
    - ▷ ***Positive values mean in favor of the root player.***
    - ▷ ***Negative values mean in favor of the opponent.***
    - ▷ ***Since it is a zero sum game, thus from the opponent's point of view, the value can be assigned $-f(p)$.***
  - **A terminal position: a position whose value can be decided.**
    - ▷ ***A position where win/loss/draw can be concluded.***
    - ▷ ***In practice, we encounter a position where some constraints, e.g., time limit and depth limit, are met.***
  - **A position p has b legal moves $p_1, p_2, \ldots, p_b$.**

Tree node numbering



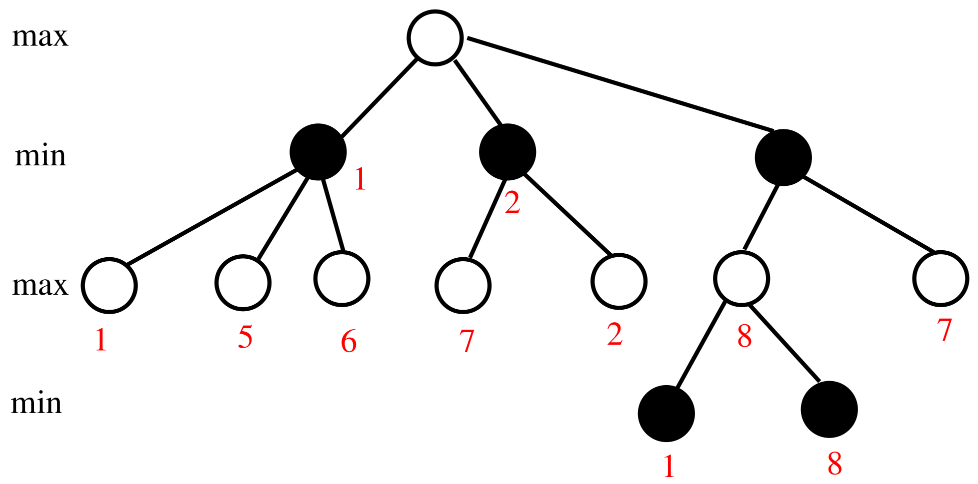
- **From the root, number a node in a search tree by a sequence of integers $a_1.a_2.a_3.a_4 \cdots$**
  - **Meaning from the root, you first take the a_1th branch, then the a_2th branch, and then the a_3th branch, and then the a_4th branch $\cdots$**
  - **The root is specified as an empty sequence.**
  - **The depth of a node is the length of the sequence of integers specifying it.**
- **This is called "Dewey decimal system."**

Mini-max formulation



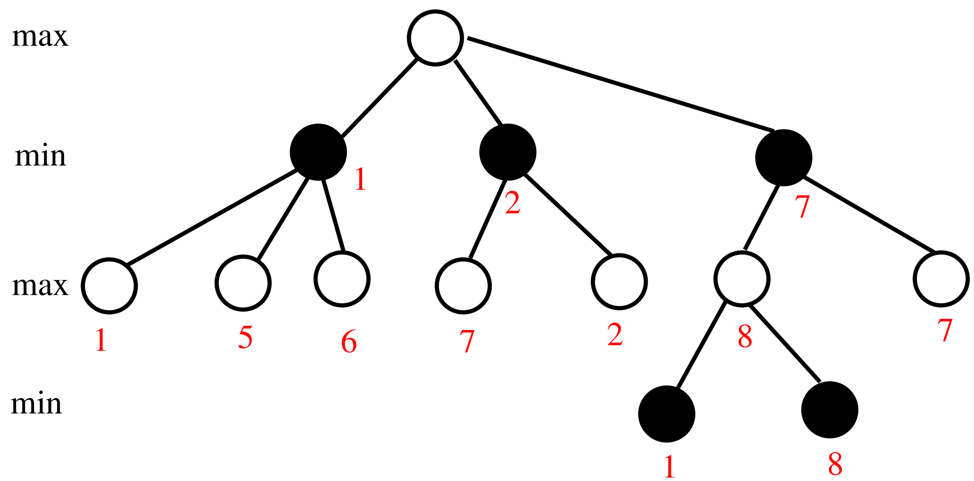
- **Mini-max formulation:**
  - $$F'(p) = \begin{cases} f(p) & \textbf{if } b = 0 \\ max\{G'(p_1), \ldots, G'(p_b)\} & \textbf{if } b > 0 \end{cases}$$
  - $$G'(p) = \begin{cases} f(p) & \textbf{if } b = 0 \\ min\{F'(p_1), \ldots, F'(p_b)\} & \textbf{if } b > 0 \end{cases}$$
  - **An indirect recursive formula with a bottom-up evaluation!**
  - **Equivalent to AND-OR logic.**

Mini-max formulation



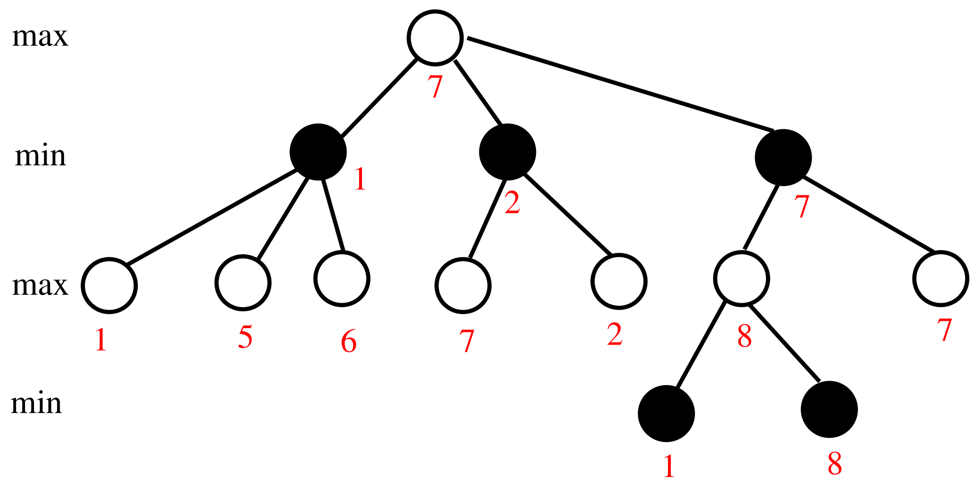
- **Mini-max formulation:**
  - $$F'(p) = \begin{cases} f(p) & \textbf{if } b = 0 \\ max\{G'(p_1), \ldots, G'(p_b)\} & \textbf{if } b > 0 \end{cases}$$
  - $$G'(p) = \begin{cases} f(p) & \textbf{if } b = 0 \\ min\{F'(p_1), \ldots, F'(p_b)\} & \textbf{if } b > 0 \end{cases}$$
  - **An indirect recursive formula with a bottom-up evaluation!**
  - **Equivalent to AND-OR logic.**

Mini-max formulation



- **Mini-max formulation:**
  - $$F'(p) = \begin{cases} f(p) & \textbf{if } b = 0 \\ max\{G'(p_1), \ldots, G'(p_b)\} & \textbf{if } b > 0 \end{cases}$$
  - $$G'(p) = \begin{cases} f(p) & \textbf{if } b = 0 \\ min\{F'(p_1), \ldots, F'(p_b)\} & \textbf{if } b > 0 \end{cases}$$
  - **An indirect recursive formula with a bottom-up evaluation!**
  - **Equivalent to AND-OR logic.**

Mini-max formulation



- **Mini-max formulation:**
  - $$F'(p) = \begin{cases} f(p) & \textbf{if } b = 0 \\ max\{G'(p_1), \ldots, G'(p_b)\} & \textbf{if } b > 0 \end{cases}$$
  - $$G'(p) = \begin{cases} f(p) & \textbf{if } b = 0 \\ min\{F'(p_1), \ldots, F'(p_b)\} & \textbf{if } b > 0 \end{cases}$$
  - **An indirect recursive formula with a bottom-up evaluation!**
  - **Equivalent to AND-OR logic.**

Algorithm: Mini-max

- **Algorithm F'(position p) // max node**
  - **determine the successor positions $p_1, \ldots, p_b$**
  - **if $b = 0$, then return $f(p)$ else begin**
    - ▷ $m := -\infty$
    - ▷ ***for*** $i := 1$ ***to*** b ***do***
    - ▷ $\quad t := G'(p_i)$
    - ▷ $\quad$ ***if*** $t > m$ ***then*** $m := t$ // ***find max value***
  - **end;**
  - **return m**
- **Algorithm G'(position p) // min node**
  - **determine the successor positions $p_1, \ldots, p_b$**
  - **if $b = 0$, then return $f(p)$ else begin**
    - ▷ $m := \infty$
    - ▷ ***for*** $i := 1$ ***to*** b ***do***
    - ▷ $\quad t := F'(p_i)$
    - ▷ $\quad$ ***if*** $t < m$ ***then*** $m := t$ // ***find min value***
  - **end;**
  - **return m**

Mini-max: comments

- **A brute-force method to try all possibilities!**
  - **May visit a position many times.**
- **Depth-first search**
  - **Move ordering is according to order the successor positions are generated.**
  - **Bottom-up evaluation.**
  - **Post-ordering traversal.**
- **Q:**
  - **Iterative deepening?**
  - **BFS?**
  - **Other types of searching?**

Mini-max: revised (1/2)

- Search a max-node position p with a depth of $depth$.
- Algorithm F'(position p, integer $depth$) // max node
  - determine the successor positions $p_1, \ldots, p_b$
  - if $b = 0$ // a terminal node
    or $depth = 0$ // remaining depth to search
    or time is running up // from timing control
    or some other constraints are met // add knowledge here
    then return $f(p)$// current board value
    else begin
      - ▷ $m := -\infty$ // *initial value*
      - ▷ ***for*** $i := 1$ ***to*** b ***do*** // *try each child*
      - ▷ ***begin***
      - ▷ $t := G'(p_i, depth - 1)$
      - ▷ ***if*** $t > m$ ***then*** $m := t$ // *find max value*
      - ▷ ***end***

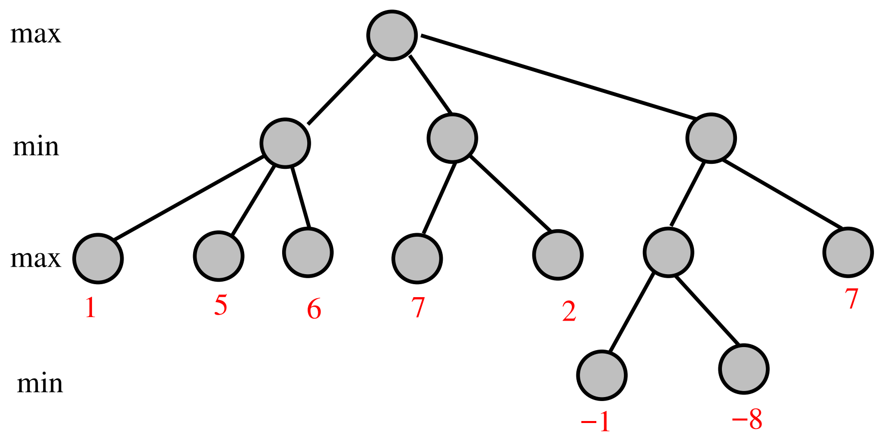
    end
  - return m

Mini-max: revised (2/2)

- **Search a min-node position p with a depth of $depth$.**
- **Algorithm G'(position p, integer $depth$) // min node**
  - **determine the successor positions $p_1, \ldots, p_b$**
  - **if $b = 0$ // a terminal node**
    or $depth = 0$ // remaining depth to search
    or time is running up // from timing control
    or some other constraints are met // add knowledge here
    then return $f(p)$// current board value
    else begin
    - ▷ $m := \infty$ *// initial value*
    - ▷ ***for*** $i := 1$ ***to*** b ***do*** *// try each child*
    - ▷ ***begin***
    - ▷ $t := F'(p_i, depth - 1)$
    - ▷   ***if*** $t < m$ ***then*** $m := t$ *// find min value*
    - ▷ ***end***

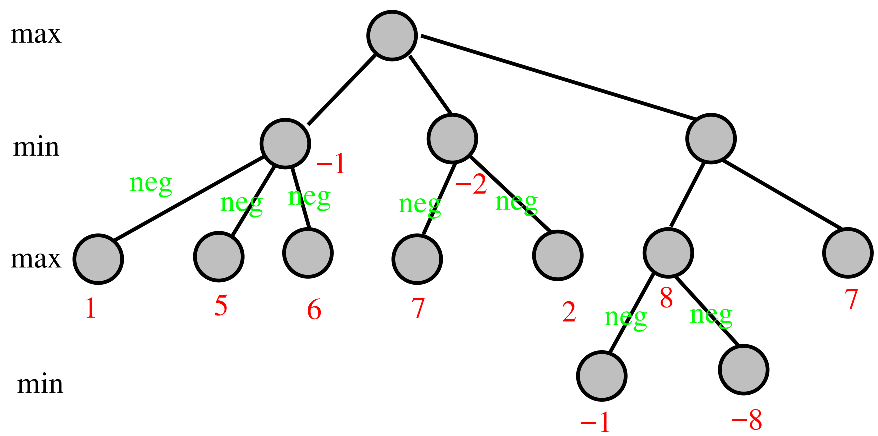
    end
  - **return m**

Nega-max formulation



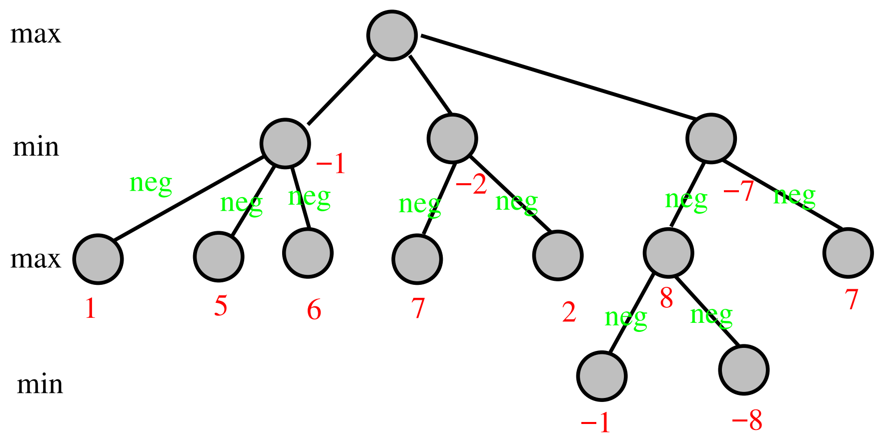
- **Nega-max formulation:**
  Let $F(p)$ be the greatest possible value achievable from position p against the optimal defensive strategy.
  - $$F(p) = \begin{cases} h(p) & \textbf{if } b = 0 \\ max\{-F(p_1), \ldots, -F(p_b)\} & \textbf{if } b > 0 \end{cases}$$
    - ▷ $$h(p) = \begin{cases} f(p) & \textit{if depth of } p \textit{ is 0 or even} \\ -f(p) & \textit{if depth of } p \textit{ is odd} \end{cases}$$
    - ▷ ***$h(p)$ is the position's value from the point of view of the player of p.***

Nega-max formulation



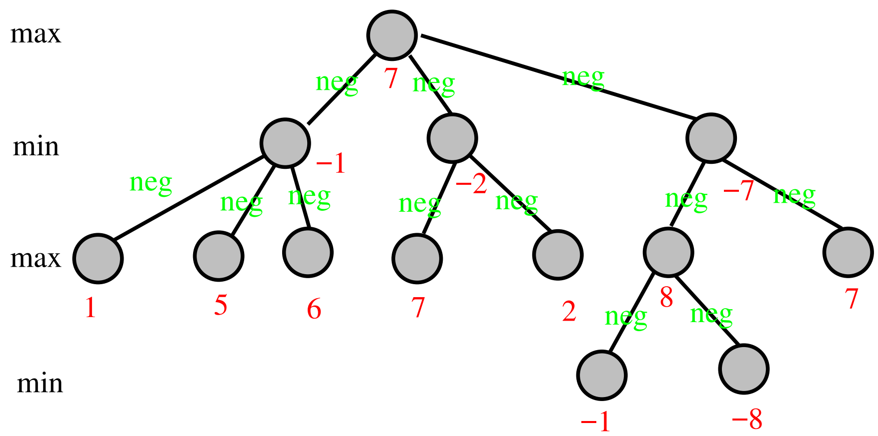
- **Nega-max formulation:**
  Let $F(p)$ be the greatest possible value achievable from position p against the optimal defensive strategy.
  - $$F(p) = \begin{cases} h(p) & \textbf{if } b = 0 \\ max\{-F(p_1), \ldots, -F(p_b)\} & \textbf{if } b > 0 \end{cases}$$
    - ▷ $$h(p) = \begin{cases} f(p) & \textit{if depth of } p \textit{ is 0 or even} \\ -f(p) & \textit{if depth of } p \textit{ is odd} \end{cases}$$
    - ▷ *$h(p)$ is the position's value from the point of view of the player of p.*

Nega-max formulation



- **Nega-max formulation:**
  Let $F(p)$ be the greatest possible value achievable from position p against the optimal defensive strategy.
  - $$F(p) = \begin{cases} h(p) & \textbf{if } b = 0 \\ max\{-F(p_1), \ldots, -F(p_b)\} & \textbf{if } b > 0 \end{cases}$$
    - ▷ $$h(p) = \begin{cases} f(p) & \textit{\textbf{if depth of }} p \textit{\textbf{ is 0 or even}} \\ -f(p) & \textit{\textbf{if depth of }} p \textit{\textbf{ is odd}} \end{cases}$$
    - ▷ ***$h(p)$ is the position's value from the point of view of the player of p.***

Nega-max formulation



- **Nega-max formulation:**
  Let $F(p)$ be the greatest possible value achievable from position p against the optimal defensive strategy.
  - $$F(p) = \begin{cases} h(p) & \textbf{if } b = 0 \\ max\{-F(p_1), \ldots, -F(p_b)\} & \textbf{if } b > 0 \end{cases}$$
    - $$h(p) = \begin{cases} f(p) & \textit{if depth of } p \textit{ is 0 or even} \\ -f(p) & \textit{if depth of } p \textit{ is odd} \end{cases}$$
    - ***$h(p)$ is the position's value from the point of view of the player of p.***

Algorithm: Nega-max

- **Algorithm F(position p, integer $depth$)**
  - determine the successor positions $p_1, \ldots, p_b$
  - if $b = 0$ // a terminal node
    or $depth = 0$ // remaining depth to search
    or time is running up // from timing control
    or some other constraints are met // add knowledge here
  - then return $h(p)$ else
  - begin
    - ▷ $m := -\infty$
    - ▷ ***for*** $i := 1$ ***to*** b ***do***
    - ▷ ***begin***
    - ▷ $t := -F(p_i, depth - 1)$ // ***recursive call, the returned value is negated***
    - ▷ ***if*** $t > m$ ***then*** $m := t$ // ***always find a max value***
    - ▷ ***end***
  - end
  - return m

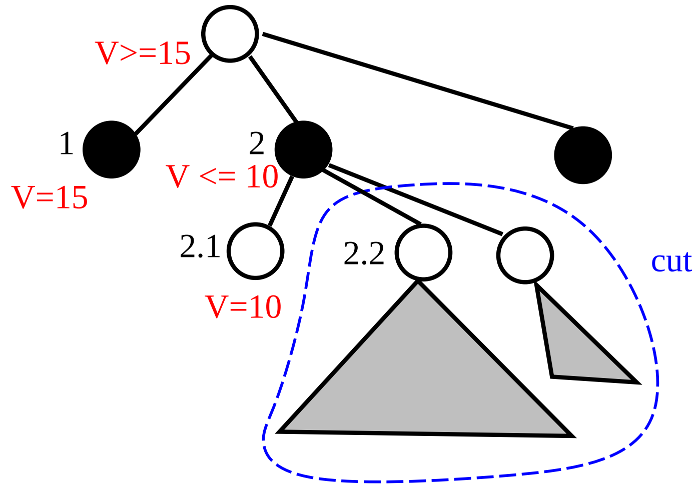
Nega-max: comments

- **Another brute-force method to try all possibilities.**
  - **Use $h(p)$ instead of $f(p)$.**
    - ▷ ***Zero-sum game: if one player thinks a position p has a value of w, then the other player thinks it is $-w$.***
    - ▷ $\min\{x, y, z\} = -max\{-x, -y, -z\}$.
    - ▷ $\max\{x, y, z\} = -min\{-x, -y, -z\}$.
  - **Watch out the code in dealing with search termination conditions.**
    - ▷ ***Leaf.***
    - ▷ ***Reach a given searching depth.***
    - ▷ ***Timing control.***
    - ▷ ***Other constraints such as the score is good or bad enough.***
- **Notations:**
  - F' **means the Mini-max version.**
    - ▷ ***Need a G' companion.***
    - ▷ ***Easy to explain.***
  - F **means the Nega-max version.**
    - ▷ ***Simpler code.***
    - ▷ ***Maybe difficult to explain.***

Intuition for improvements

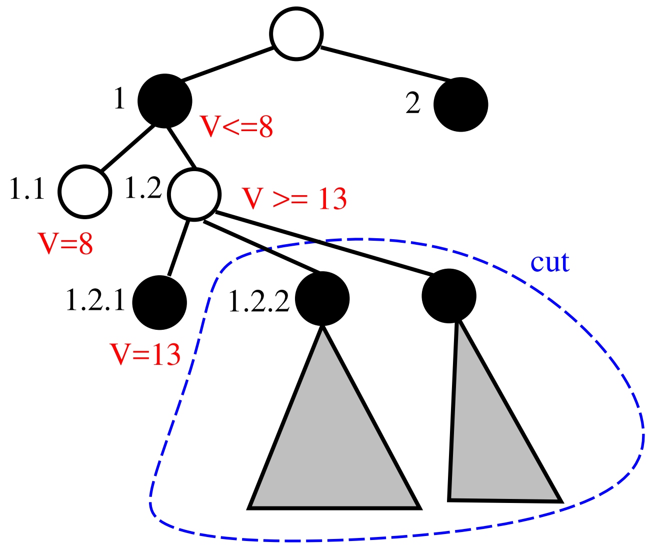
- **Branch-and-bound: using information you have so far to cut or prune branches.**
  - **A branch is cut means we do not need to search it anymore.**
  - **If you know for sure or almost sure the value of your result is more than x and the current search result for this branch so far can give you no more than x,**
    - ▷ *then there is no need to search this branch any further.*
- **Two types of approaches**
  - **Exact algorithms: through mathematical proof, it is guaranteed that the branches pruned won't contain the solution.**
    - ▷ *Alpha-beta pruning: reinvented by several researchers in the 1950's and 1960's.*
    - ▷ *Scout.*
    - ▷ $\cdots$
  - **Approximated heuristics: with a high probability that the solution won't be contained in the branches pruned.**
    - ▷ *Obtain a good estimation on the remaining cost.*
    - ▷ *Cut a branch when it is in a very bad position and there is little hope to gain back the advantage.*

Alpha cut-off



- **On the max node which is the root:**
  - ▷ ***Assume you have finished exploring the branch at*** 1 ***and obtained the best value from it as*** $bound$.
  - ▷ ***You now search the branch at*** 2 ***by first searching the branch at*** 2.1.
  - ▷ ***Assume branch at*** 2.1 ***returns a value that is*** $\leq bound$.
  - ▷ ***Then no need to evaluate the branch at*** 2.2 ***and all later branches of*** 2, ***if any, at all.***
  - ▷ ***The best possible value for the branch at*** 2 ***must be*** $\leq bound$.
  - ▷ ***Hence we should take value returned from the branch at*** 1 ***as the best possible solution.***

Beta cut-off

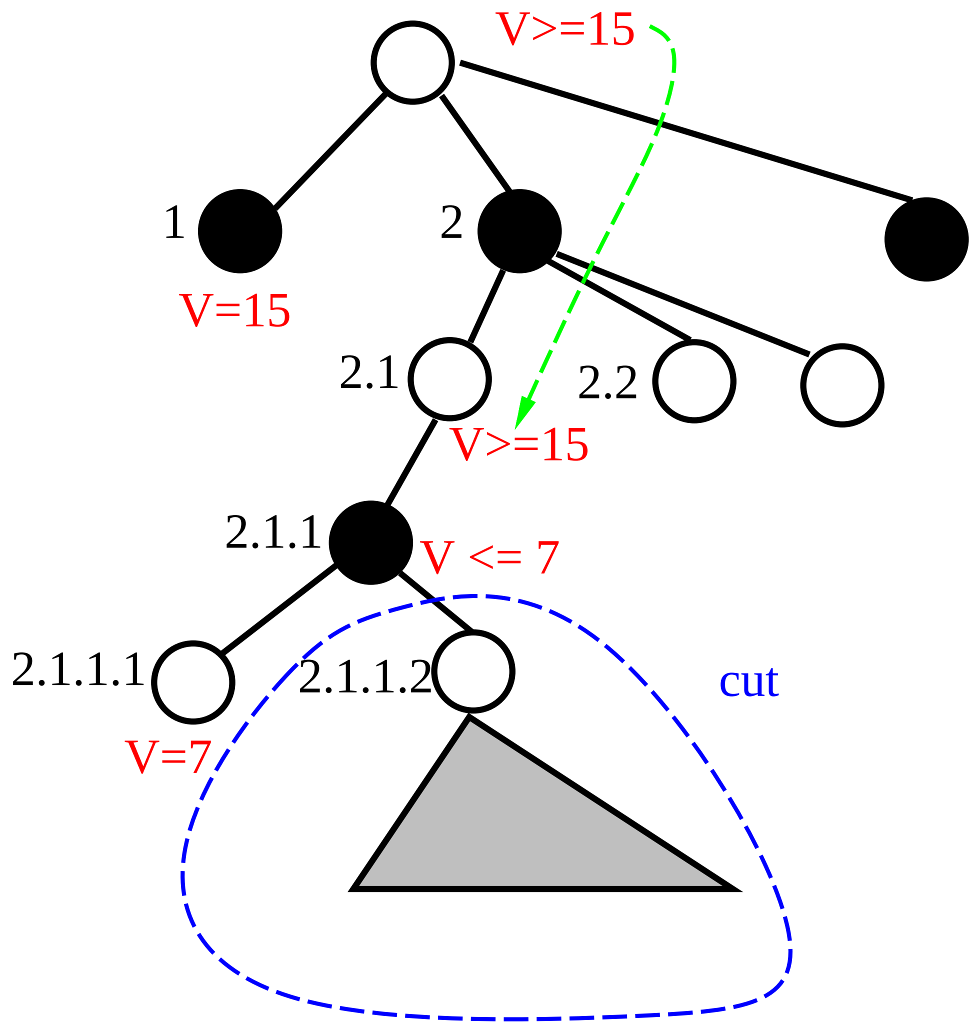


- **On the min node 1:**
  - ***Assume you have finished exploring the branch at 1.1 and obtained the best value from it as*** $bound$.
  - ***You now search the branch at 1.2 by first exploring the branch at 1.2.1.***
  - ***Assume the branch at 1.2.1 returns a value that is*** $\geq bound$.
  - ***Then no need to evaluate the branch at 1.2.2 and all later branches of 1.2, if any, at all.***
  - ***The best possible value for the branch at 1.2 is*** $\geq bound$.
  - ***Hence we should take value returned from the branch at 1.1 as the best possible solution.***

Deep alpha cut-off

- **For alpha cut-off:**
  - ▷ ***For a min node u, a branch of its ancestor (e.g., an elder brother of its parent) produces a lower bound V_l.***
  - ▷ ***The first branch of u produces an upper bound V_u for v.***
  - ▷ ***If $V_l \geq V_u$, then there is no need to evaluate the second branch and all later branches, of u.***
- **Deep alpha cut-off:**
  - ▷ ***DEF: For a node u in a tree and a positive integer g, Ancestor(g, u) is the direct ancestor of u by tracing the parent's link g times.***
  - ▷ ***When the lower bound V_l is produced at and propagated from u's great grand parent, i.e., Ancestor(3,u), or any Ancestor($2i+1$,u), $i \geq 1$.***
  - ▷ ***When an upper bound V_u is returned from the a branch of u and $V_l \geq V_u$, then there is no need to evaluate all later branches of u.***
- **We can find similar properties for deep beta cut-off.**

Illustration — Deep alpha cut-off



Ideas for refinements

- **During searching, maintain two values $alpha$ and $beta$ so that**
  - $alpha$ **is the current lower bound of the possible returned value;**
    - ▷ *This means to say you know a way to achieve the value $alpha$.*
  - $beta$ **is the current upper bound of the possible returned value.**
    - ▷ *This means to say your opponent knows a way to achieve a value of $beta$.*
  - **If $alpha = beta$, then we have found the solution.**
- **If during searching, we know for sure $alpha > beta$, then there is no need to search any more in this branch.**
  - **The returned value cannot be in this branch.**
  - **Backtrack until it is the case $alpha \leq beta$.**
- **The two values $alpha$ and $beta$ are called the ranges of the current search window.**
  - **These values are dynamic.**
  - **Initially, $alpha$ is $-\infty$ and $beta$ is ∞.**

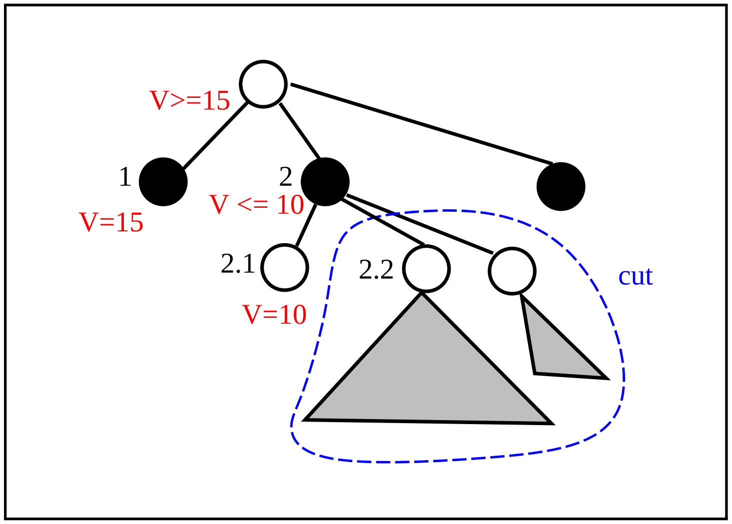
Alpha-beta pruning algorithm: Mini-Max

- **Algorithm $F1'$(position p, value $alpha$, value $beta$) // max node**
  - **determine the successor positions $p_1, \ldots, p_b$**
  - **if $b = 0$, then return $f(p)$ else begin**
    - ▷ $m := alpha$
    - ▷ ***for*** $i := 1$ ***to*** b ***do***
    - ▷ $\quad t := G1'(p_i, m, beta)$
    - ▷ $\quad$ ***if*** $t > m$ ***then*** $m := t$ ***// improve the current best value***
    - ▷ $\quad$ ***if*** $m \geq beta$ ***then return***($beta$) ***// beta cut off***
  - **end; return m**
- **Algorithm $G1'$(position p, value $alpha$, value $beta$) // min node**
  - **determine the successor positions $p_1, \ldots, p_b$**
  - **if $b = 0$, then return $f(p)$ else begin**
    - ▷ $m := beta$
    - ▷ ***for*** $i := 1$ ***to*** b ***do***
    - ▷ $\quad t := F1'(p_i, alpha, m)$
    - ▷ $\quad$ ***if*** $t < m$ ***then*** $m := t$
    - ▷ $\quad$ ***if*** $m \leq alpha$ ***then return***($alpha$) ***// alpha cut off***
  - **end; return m**

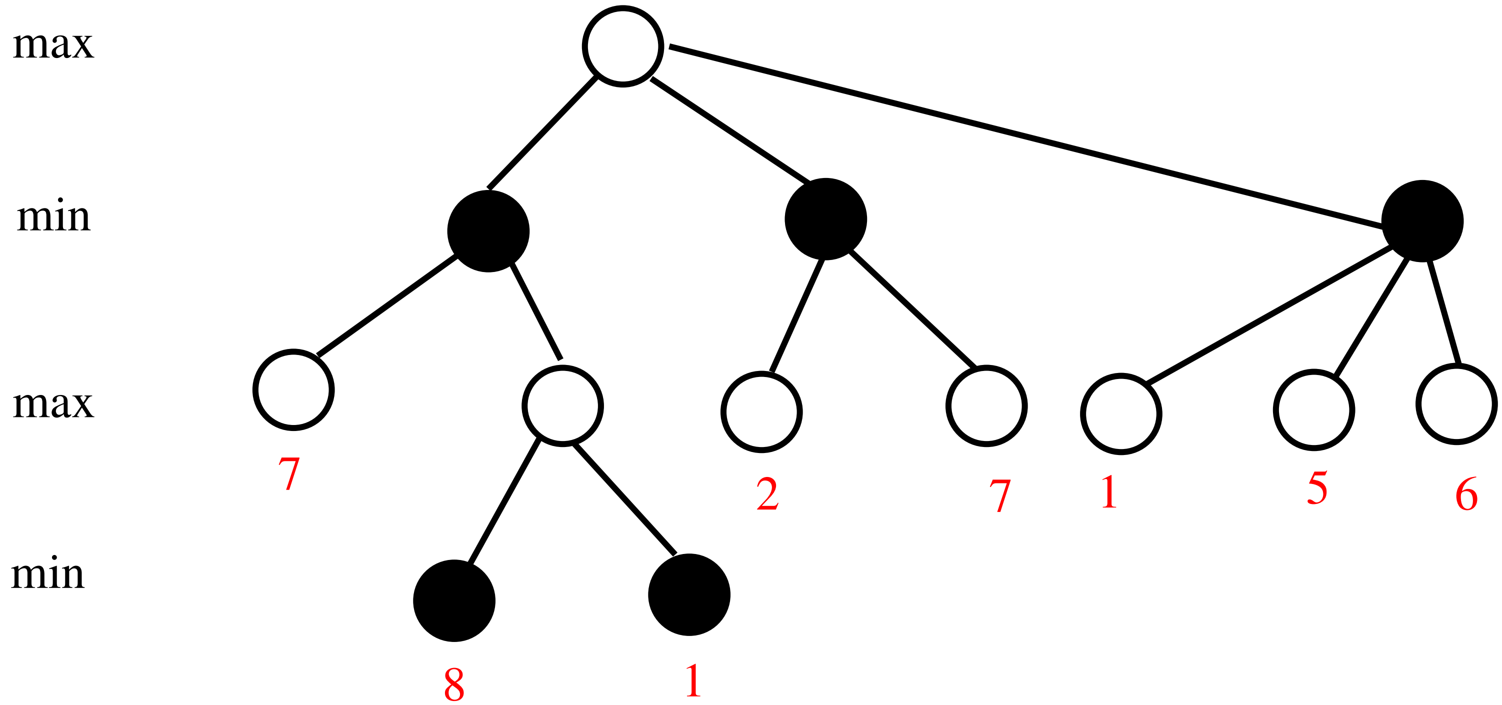
Example

Initial call: $F1'$(root,$-\infty$,∞)

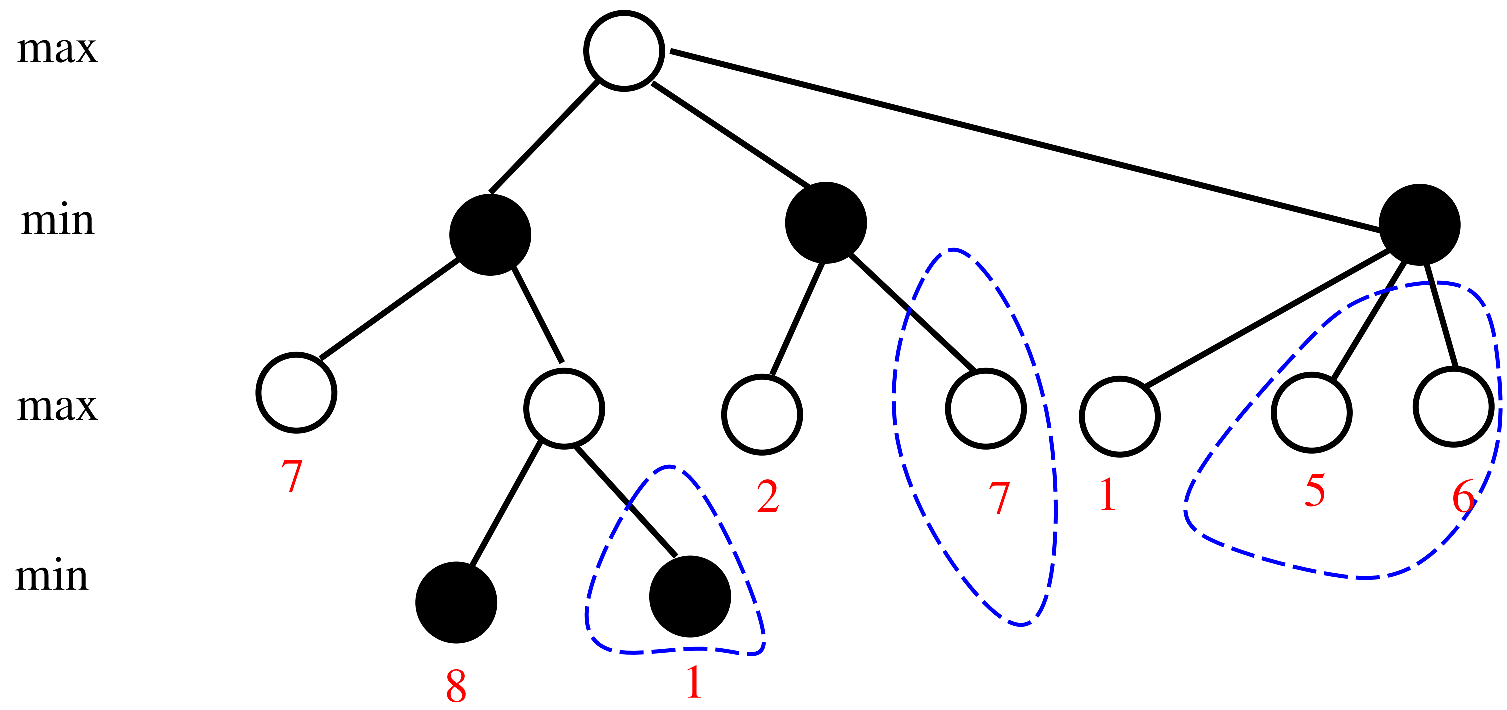
- $m = -\infty$
- **call** $G1'$**(node 1,**$-\infty$**,**∞**)**
  - ▷ ***it is a terminal node***
  - ▷ ***return value*** 15
- $t = 15$**;**
  - ▷ ***since*** $t > m$***,*** m ***is now*** 15
- **call** $G1'$**(node 2,**15**,**∞**)**
  - ▷ ***call*** $F1'$***(node 2.1,***15***,***∞***)***
  - ▷ ***it is a terminal node; return*** 10
  - ▷ $t = 10$***; since*** $t < \infty$***,*** m ***is now*** 10
  - ▷ $alpha$ ***is*** 15***,*** m ***is*** 10***, so we have an alpha cut off,***
  - ▷ ***no need to call*** $F1'$***(node 2.2,15,10)***
  - ▷ ***return 15***
  - ▷ $\cdots$



A complete example



A complete example

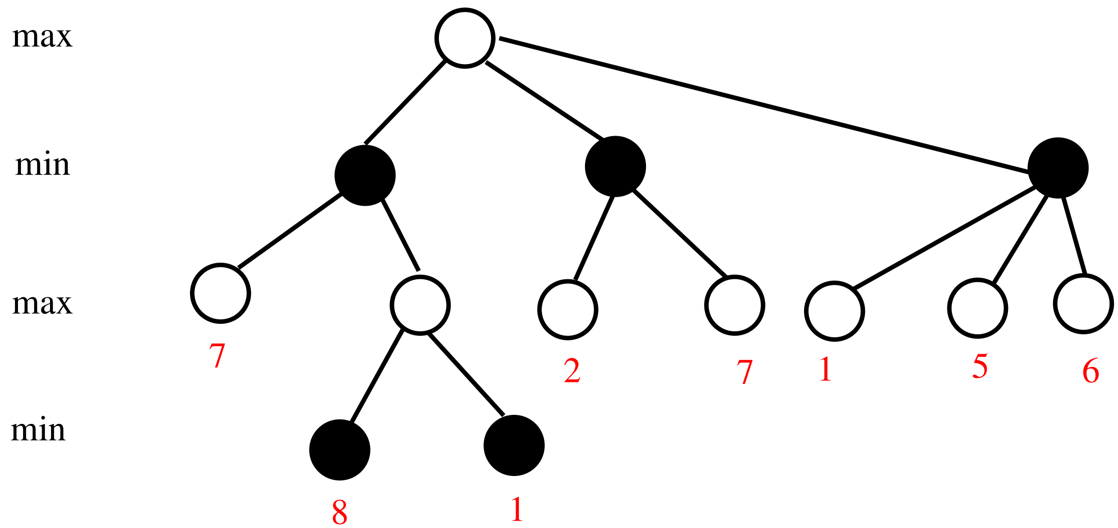


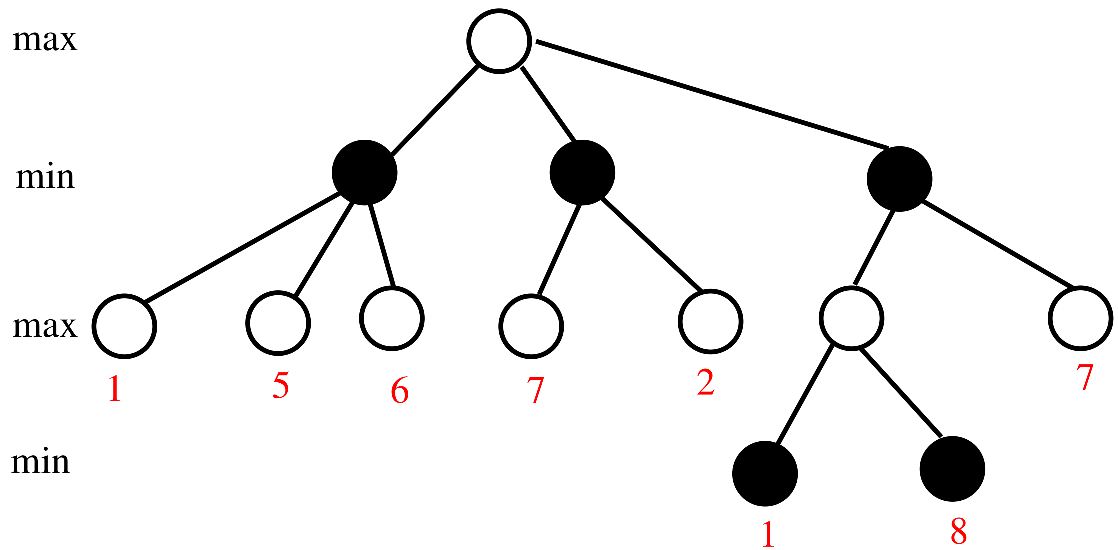
- **The solution is the same with or without the cut.**

Alpha-beta pruning algorithm: Nega-max

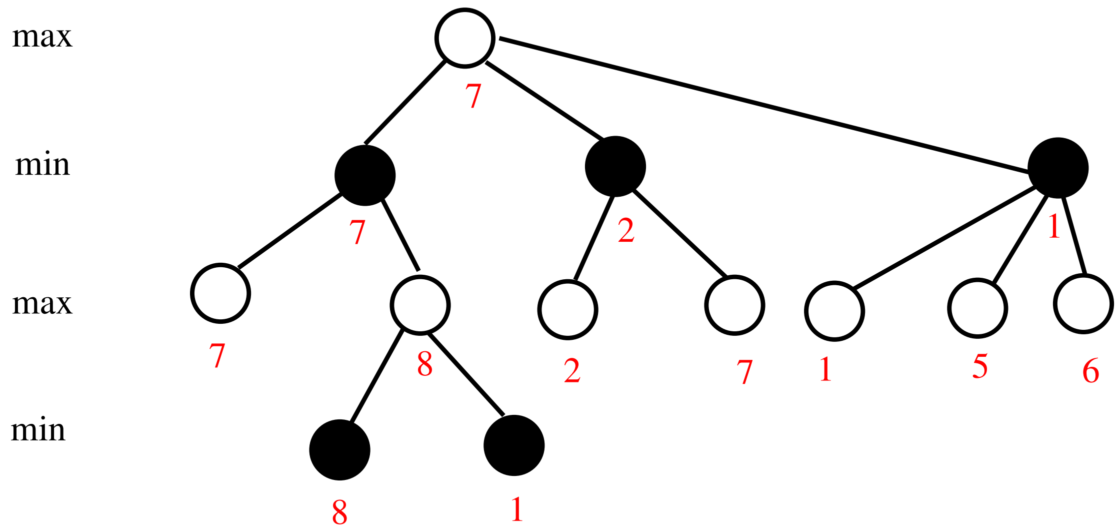
- **Algorithm** $F1$**(position** p**, value** $alpha$**, value** $beta$**, integer** $depth$**)**
  - **determine the successor positions** $p_1, \ldots, p_b$
  - **if** $b = 0$ **// a terminal node**
    or $depth = 0$ **// remaining depth to search**
    or time is running up // from timing control
    or some other constraints are met // add knowledge here
  - **then return** $h(p)$ **else**
  - **begin**
    - ▷ $m := alpha$
    - ▷ ***for*** $i := 1$ ***to*** b ***do***
    - ▷ ***begin***
    - ▷ $t := -F1(p_i, -beta, -m, depth - 1)$
    - ▷ ***if*** $t > m$ ***then*** $m := t$
    - ▷ ***if*** $m \geq beta$ ***then return***$(beta)$ ***// cut off***
    - ▷ ***end***
  - **end**
  - **return** m

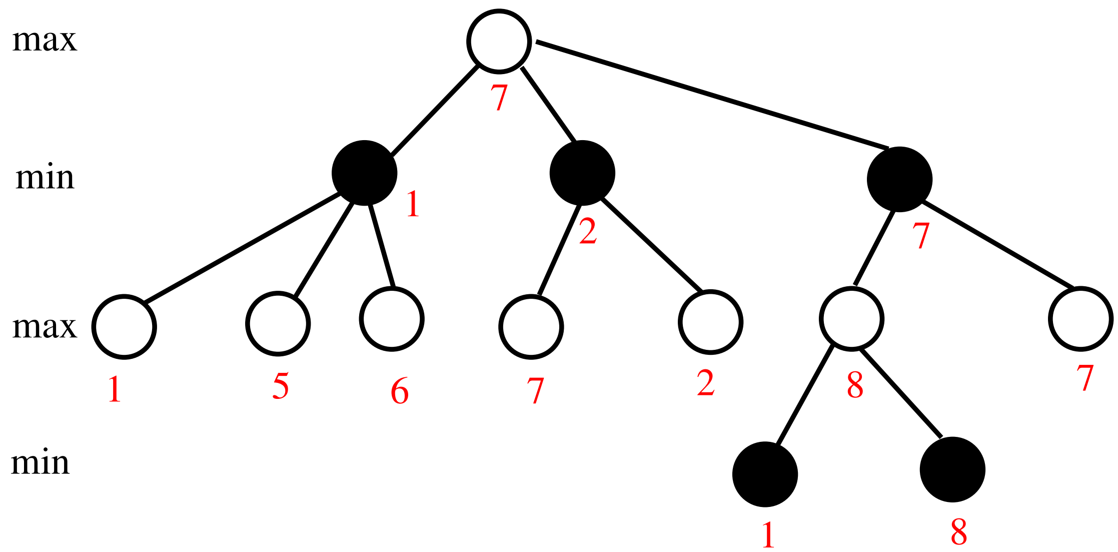
Examples (1/4)



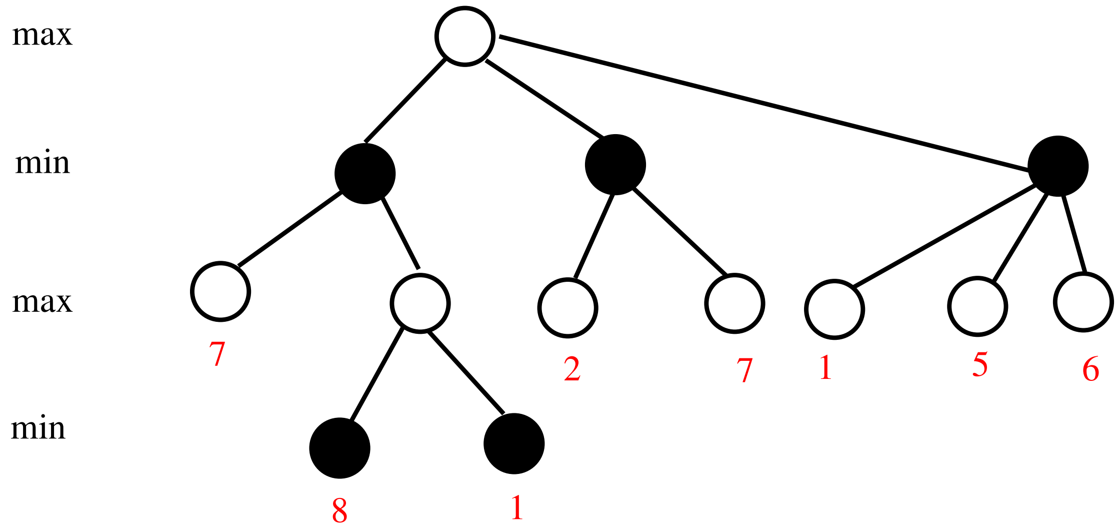


Examples (2/4)

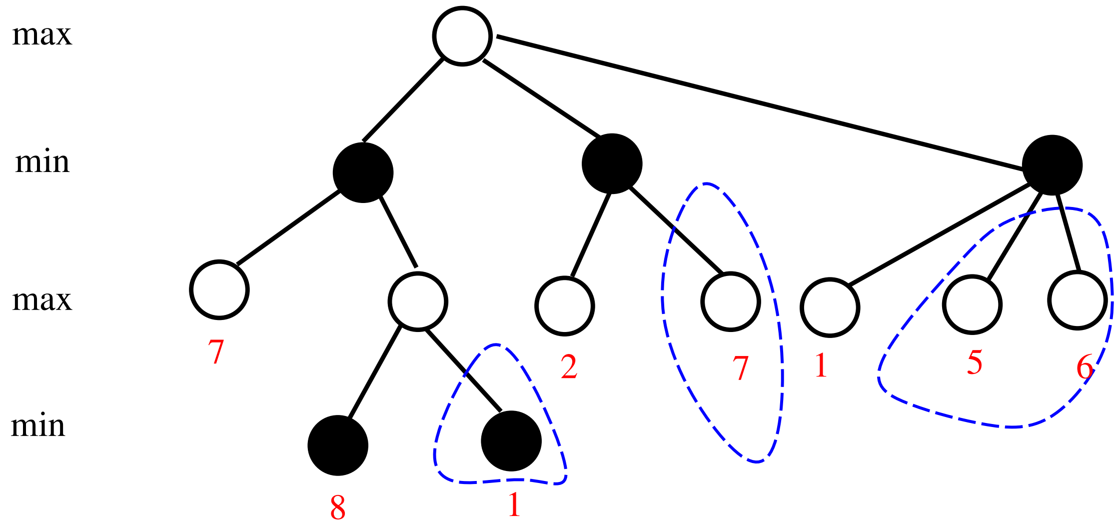

max
min
max
min
7
7
2
1
7
8
2
7
1
5
6
8
1


max
min
max
min
7
1
2
7
1
5
6
7
2
8
7
1
8

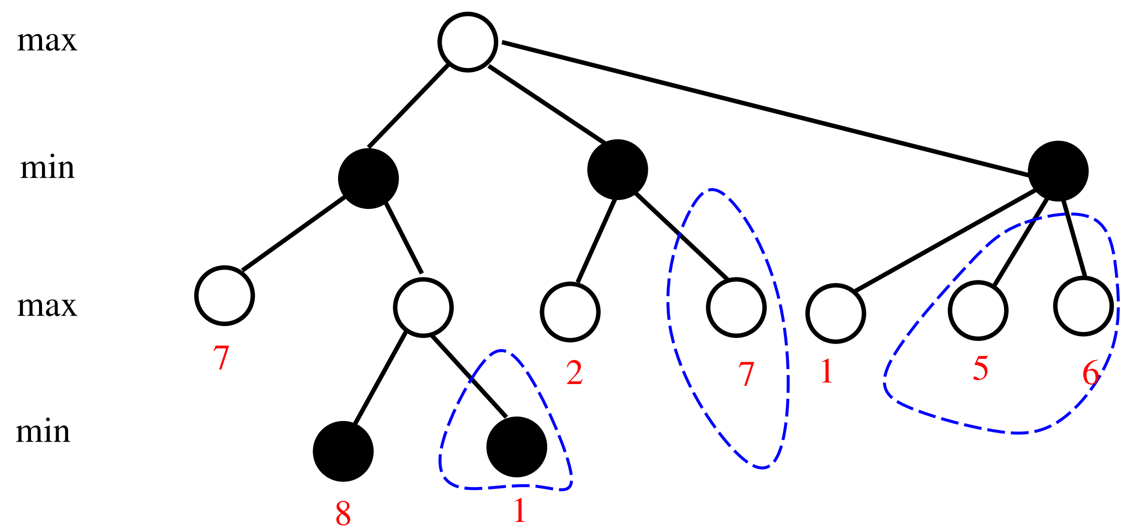
Examples (3/4)

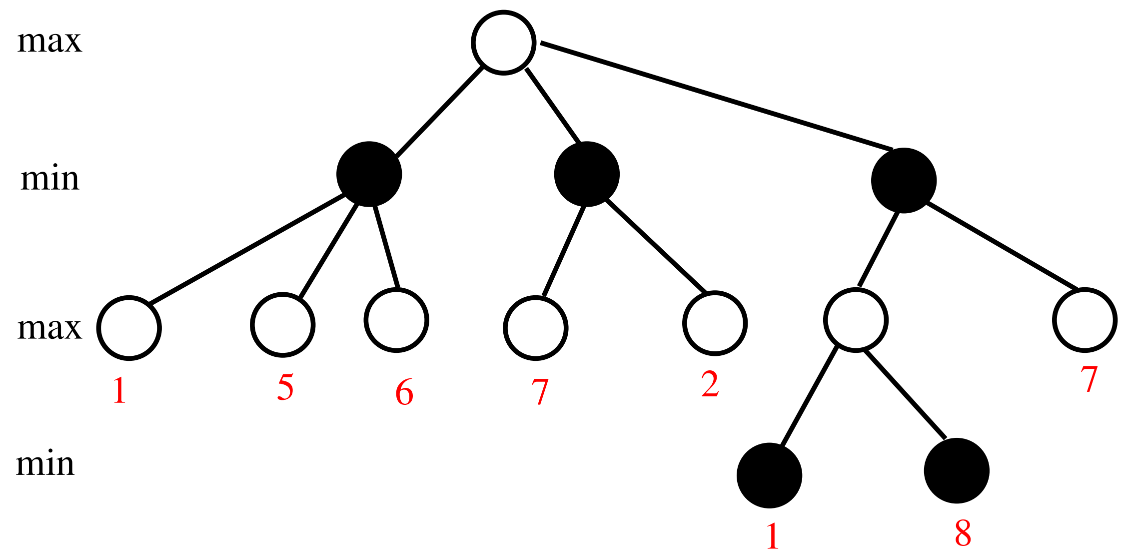


Examples (3/4)



Examples (4/4)





Lessons from the previous examples

- It looks like for the same tree, **different move orderings** give very different cut branches.
- It looks like **if a node can evaluate a child with the best possible outcome earlier**, then it has a chance to cut earlier.
  - For a min node, this means **to search the child branch that gives the lowest value first.**
  - For a max node, this means **to search the child branch that gives the highest value first.**
- Comments:
  - Watch out **the returned value** when alpha or beta cut-off happens.
    - ▷ *It is the value of one of the current window bound, obtained in other branches, not the one in the current branch.*
  - It is impossible to always know which the best branch is; otherwise we do not have to do a brute-force search.
  - Q: In the best case scenario, how many nodes can be cut?

Analysis of a possible best case

- **Definitions:**
  - **A path in a search tree is a sequence of numbers indicating the branches selected in each level using the Dewey decimal system.**
  - **A position is denoted as a path $a_1.a_2.\cdots.a_\ell$ from the root.**
  - **A position $a_1.a_2.\cdots.a_\ell$ is critical if**
    - ▷ *$a_i = 1$ for all even values of i or*
    - ▷ *$a_i = 1$ for all odd values of i.*
  - **Note: as a special case, the root is critical.**
  - **Examples:**
    - ▷ *2.1.4.1.2, 1.3.1.5.1.2, 1.1.1.2.1.1.1.3 and 1.1 are critical*
    - ▷ *1.2.1.1.2 is not critical*
  - **The number of 1's in a path has little to do with whether it is critical or not.**
- **Q: Why does the root need to be critical?**

Perfect-ordering tree

- **A perfect-ordering tree:**

$$F(a_1.\cdots.a_\ell) = \begin{cases} h(a_1.\cdots.a_\ell) & \textbf{if } a_1.\cdots.a_\ell \textbf{ is a terminal} \\ -F(a_1.\cdots.a_\ell.1) & \textbf{otherwise} \end{cases}$$

  - **The first successor of every non-terminal position gives the best possible value.**

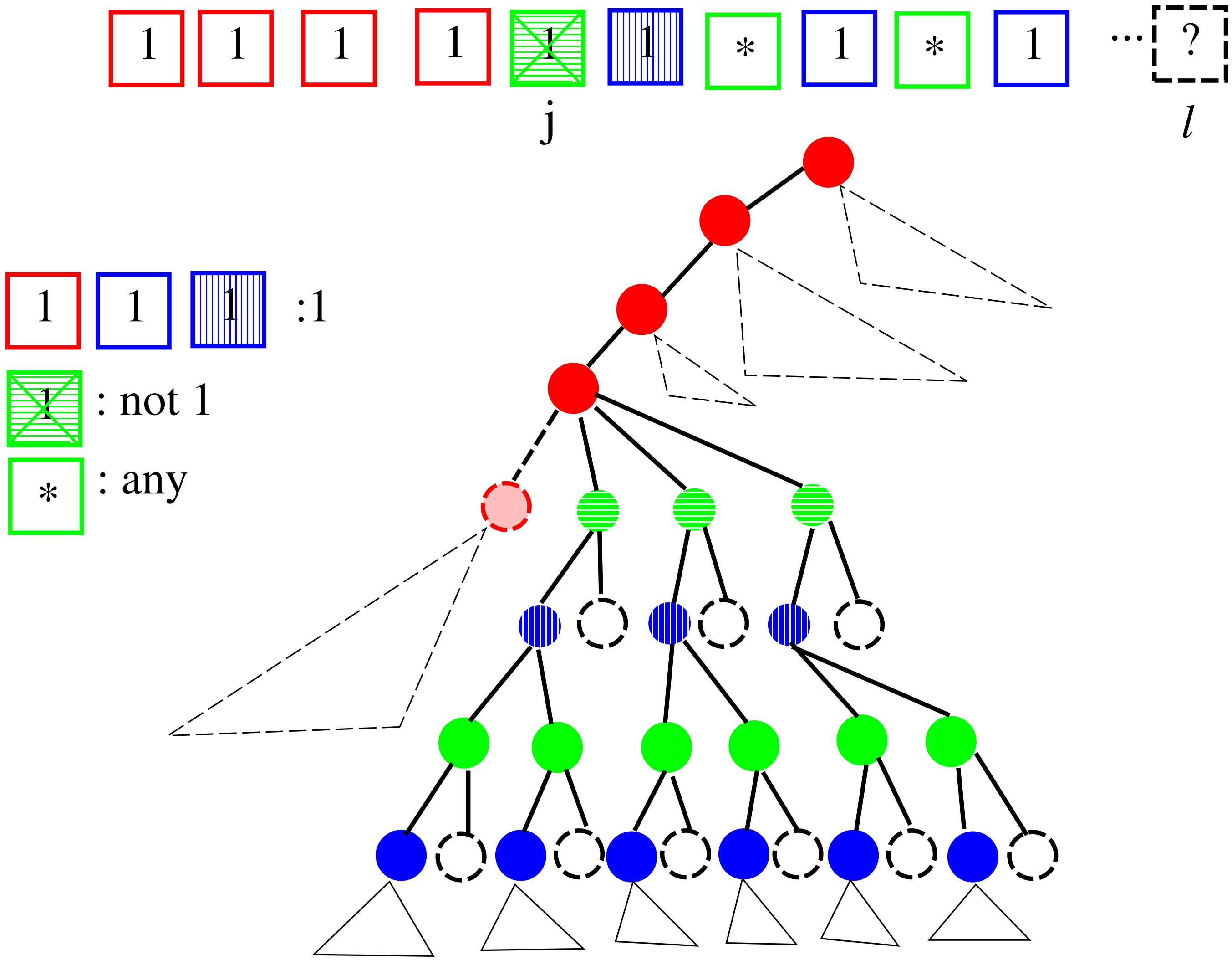
Theorem 1

- **Theorem 1: $F1$ examines precisely the critical positions of a perfect-ordering tree.**
- **Proof sketch:**
  - **Classify the critical positions, a.k.a. nodes, into different types.**
    - ▷ ***You must evaluate the first branch from the root to the bottom.***
    - ▷ ***Alpha cut off happens at odd-depth nodes as soon as the first branch of this node is evaluated.***
    - ▷ ***Beta cut off happens at even-depth nodes as soon as the first branch of this node is evaluated.***
  - **For nodes of the same type, associate them with pruning of same characteristics occurred.**

Types of nodes

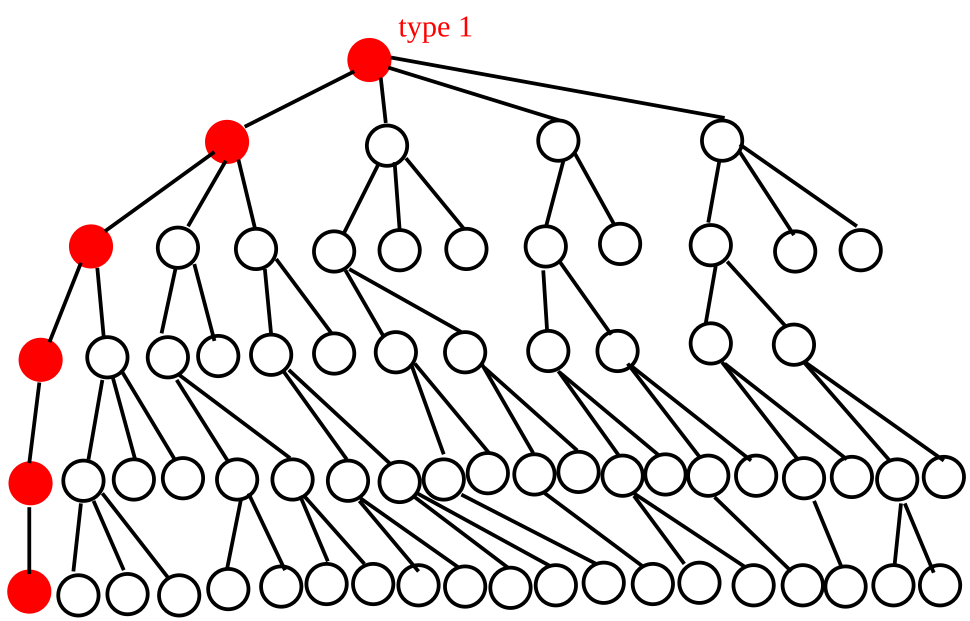
- **Classification of critical positions $a_1.a_2.\cdots.a_j.\cdots.a_\ell$ where j is the least index, if exists, such that $a_j \neq 1$ and ℓ is the last index.**
  - **j is the anchor in the analysis.**
  - **DEF: let $IS1(a_i)$ be a boolean function so that it is 0 if it is not the value 1 and it is 1 if it is.**
    - ▷ ***We call this $IS1$ parity of a number.***
  - **If j exists and $\ell > j$, then**
    - ▷ $a_{j+1} = 1$ ***because this position is critical and thus the $IS1$ parities of a_j and a_{j+1} are different.***
  - **Since this position is critical, if $a_j \neq 1$, then $a_h = 1$ for any h such that $h - j$ is odd.**
- **We now classify critical nodes into three types.**
  - **Nodes of the same type share some common properties.**

Illustration — critical nodes



Type 1 nodes

- **type 1: the root, or a node with all the a_i are 1;**
  - **This means j does not exist.**
  - **Nodes on the leftmost branch.**
  - **The leftmost child of a type 1 node except the root.**
- **In a DFS-like searching, type 1 nodes are examined first.**



Type 2 nodes

- **Classification of critical positions $a_1.a_2.\cdots.a_j.\cdots.a_\ell$ where j is the least index such that $a_j \neq 1$ and ℓ is the last index.**
- **The anchor j exists.**
- **Type 2: $\ell - j$ is zero or even;**
  - **type 2.1: $\ell - j = 0$ which means $\ell = j$.**
    - ▷ ***It is in the form of*** $\underline{1.1.1.\cdots.1.1.1}.a_\ell$ ***and*** $a_\ell \neq 1$.
    - ▷ ***The non-leftmost children of a type 1 node.***
  - **type 2.2: $\ell - j > 0$ and is even.**
    - ▷ ***It is in the form of*** $\underline{1.1.\cdots.1.1.a_j.1.a_{j+2}.\cdots.a_{\ell-2}.1}.a_\ell$.
    - ▷ ***Note, we have already defined*** $1.1.\cdots.1.1.a_j.1.a_{j+2}.\cdots.a_{\ell-2}.1$ ***to be a type 3 node.***
    - ▷ ***All of the children of a type 3 node.***
- **Q:**
  - **Can a_ℓ be 1 or non-1 for a type 2 node?**
  - **Can a_ℓ be 1 or non-1 for a type 2.1 node?**
  - **Can a_ℓ be 1 or non-1 for a type 2.2 node?**

Type 3 nodes

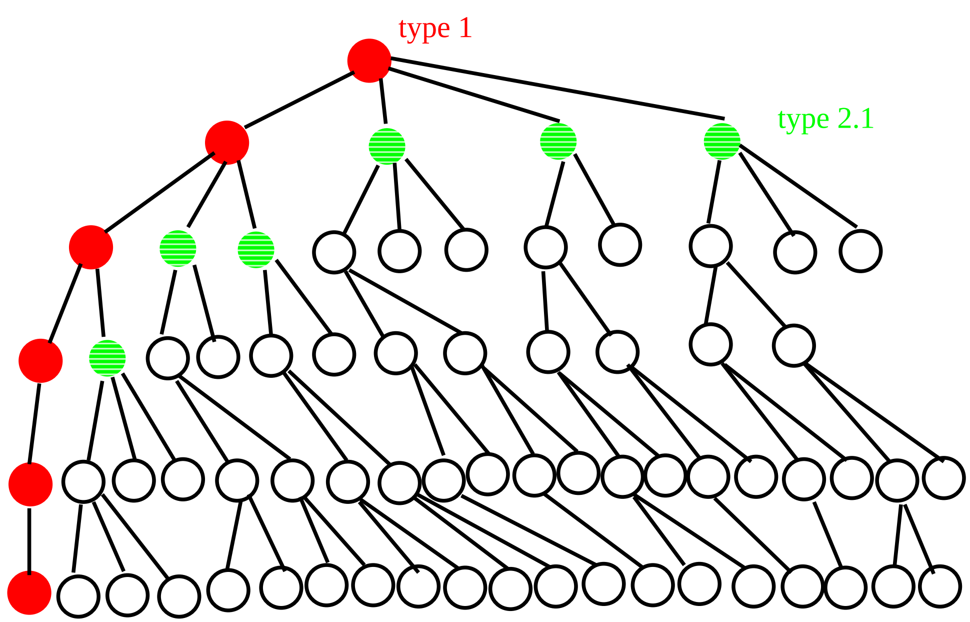
- **Classification of critical positions $a_1.a_2.\cdots.a_j.\cdots.a_\ell$ where j is the least index such that $a_j \neq 1$ and ℓ is the last index.**
- **The anchor j exists.**
- **Type 3: $\ell - j$ is odd;**
  - $a_j \neq 1$ and $\ell - j$ is odd
    - ▷ ***Since this position is critical, the $IS1$ parities of a_j and a_ℓ are different.***
      $\Longrightarrow a_\ell = 1$
      $\Longrightarrow a_{j+1} = 1$
  - **It is in the form of**
    - ▷ $1.1.\cdots.1.a_j.1.a_{j+2}.1.\cdots.1.a_{\ell-1}.1.$
  - **The leftmost child of a type 2 node.**
  - **type 3.1: $\ell - j = 1$.**
    - ▷ ***It is of the form*** $\underline{1.1.\cdots.1.a_j}.1$
    - ▷ ***The leftmost child of a type 2.1 node.***
  - **type 3.2: $\ell - j > 1$.**
    - ▷ ***It is of the form*** $\underline{1.1.\cdots.1.a_j.1.a_{j+2}.1.\cdots.1.a_{\ell-1}}.1$
    - ▷ ***The leftmost child of a type 2.2 node.***
- **Q: Can a_ℓ be 1 or non-1 for a type 3 node?**

Comments

- **Nodes of the same type have common properties.**
- **These properties can be used in solving other problems.**
  - **Example: Efficient parallelization of alpha-beta based searching algorithms.**
- **Main techniques used:**
  - **For each non-1 number, any number appeared later and is odd distance away must be 1.**
    - ▷ ***You cannot have two consecutive non-1 numbers in the ID of a critical node.***

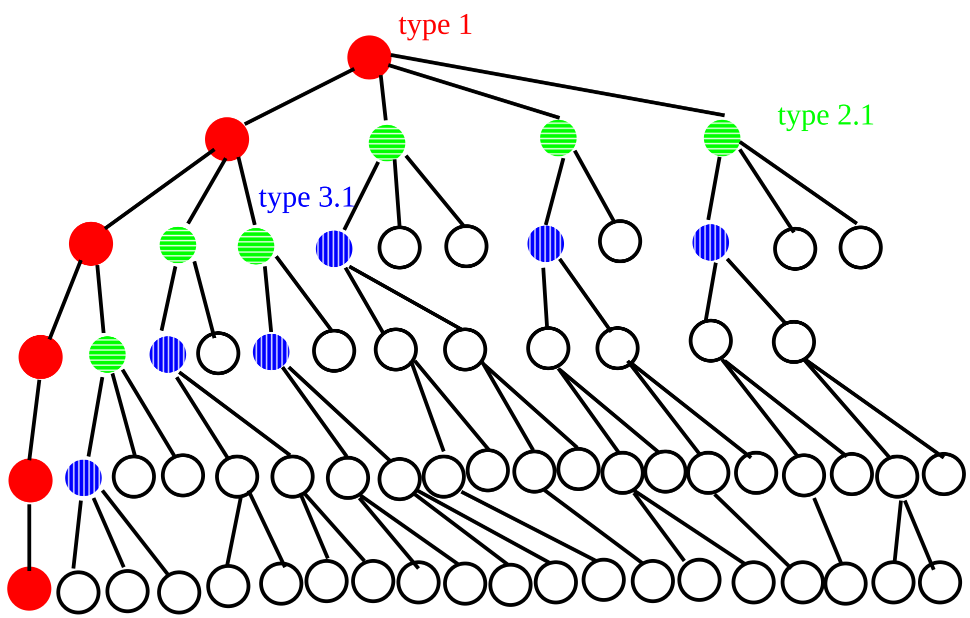
Type 2.1 nodes

- **Classification of critical positions $a_1.a_2.\cdots.a_j.\cdots.a_\ell$ where j is the least index such that $a_j \neq 1$ and ℓ is the last index.**
- **type 2: $\ell - j$ is zero or even;**
  - **type 2.1: $\ell - j = 0$.**
    - ▷ ***Then $\ell = j$.***
    - ▷ ***It is in the form of $\underline{1.1.1.\cdots.1.1.1}.a_\ell$ and $a_\ell \neq 1$.***
    - ▷ ***The non-leftmost children of a type 1 node.***



Type 3.1 nodes

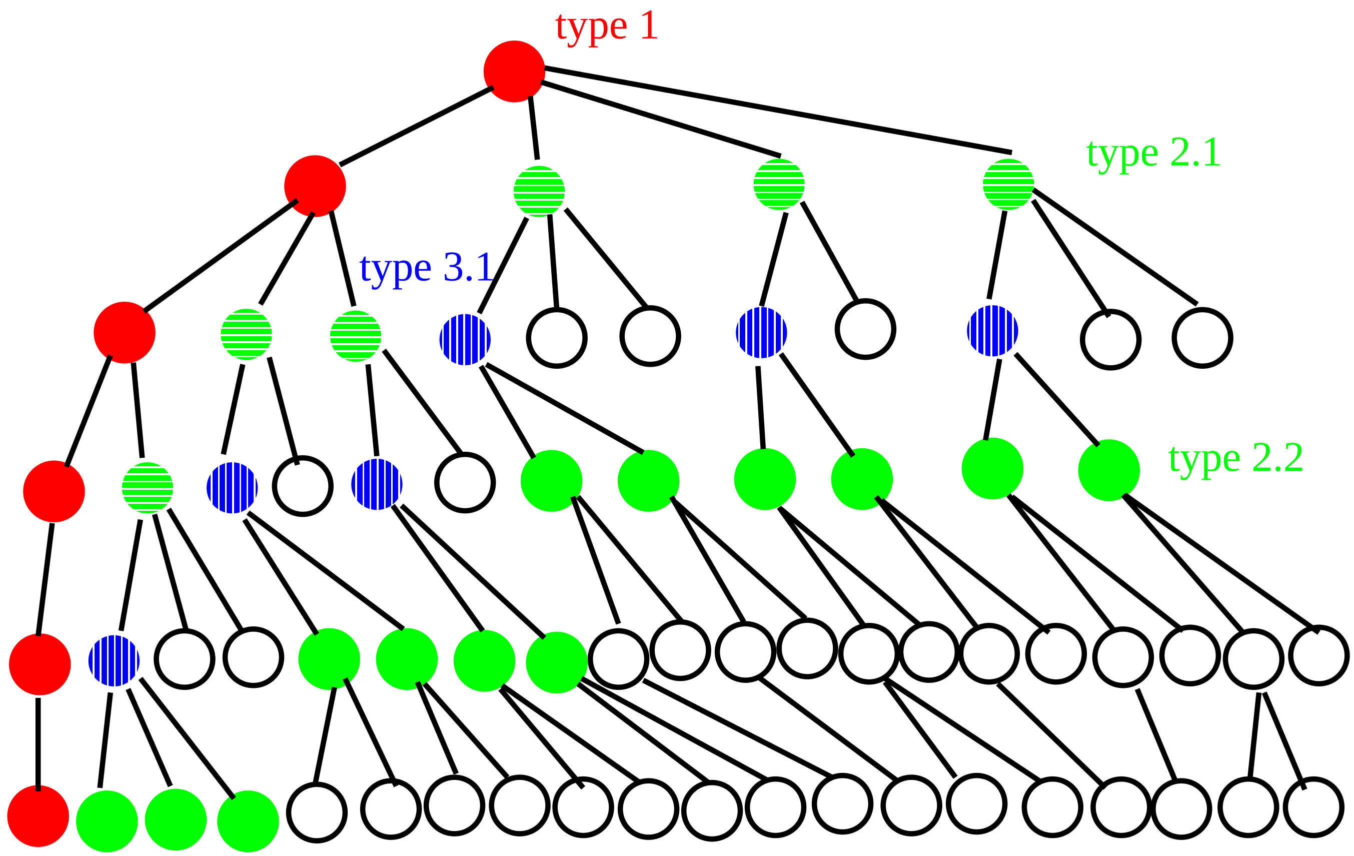
- **Classification of critical positions $a_1.a_2.\cdots.a_j.\cdots.a_\ell$ where j is the least index such that $a_j \neq 1$ and ℓ is the last index.**
- **type 3: $\ell - j$ is odd;**
  - **type 3.1: $\ell - j = 1$.**
    - ▷ ***It is of the form $\underline{1.1.\cdots.1.a_j}.1$ and $a_\ell \neq 1$.***
    - ▷ ***The leftmost child of a type 2.1 node.***



Type 2.2 nodes

- **Classification of critical positions $a_1.a_2.\cdots.a_j.\cdots.a_\ell$ where j is the least index such that $a_j \neq 1$ and ℓ is the last index.**
- **type 2: $\ell - j$ is zero or even;**
  - **type 2.2: $\ell - j > 0$ and is even.**
    - ▷ ***The $IS1$ parties of a_j and a_{j+1} are different.***
      $\Longrightarrow$ ***Since $a_j \neq 1$, $a_{j+1} = 1$.***
    - ▷ ***$(\ell - 1) - j$ is odd:***
      $\Longrightarrow$ ***The $IS1$ parties of $a_{\ell-1}$ and a_j are different.***
      $\Longrightarrow$ ***Since $a_j \neq 1$, $a_{\ell-1} = 1$.***
    - ▷ ***It is in the form of $\underline{1.1.\cdots.1.1.a_j.1.a_{j+2}.\cdots.a_{\ell-2}.1}.a_\ell$.***
    - ▷ ***Note, we will show $1.1.\cdots.1.1.a_j.1.a_{j+2}.\cdots.a_{\ell-2}.1$ is a type 3 node later.***
    - ▷ ***All of the children of a type 3 node.***

Illustration: Type 2.2 nodes



Type 3.2 nodes

- **Classification of critical positions $a_1.a_2.\cdots.a_j.\cdots.a_\ell$ where j is the least index such that $a_j \neq 1$ and ℓ is the last index.**
- **type 3: $\ell - j$ is odd;**
  - **type 3.2: $\ell - j > 1$.**
    - ▷ ***It is of the form*** $\underline{1.1.\cdots.1.a_j.1.a_{j+2}.1.\cdots.1.a_{\ell-1}}.1$
    - ▷ ***The leftmost child of a type 2.2 node.***

Illustration: Type 3.2 nodes

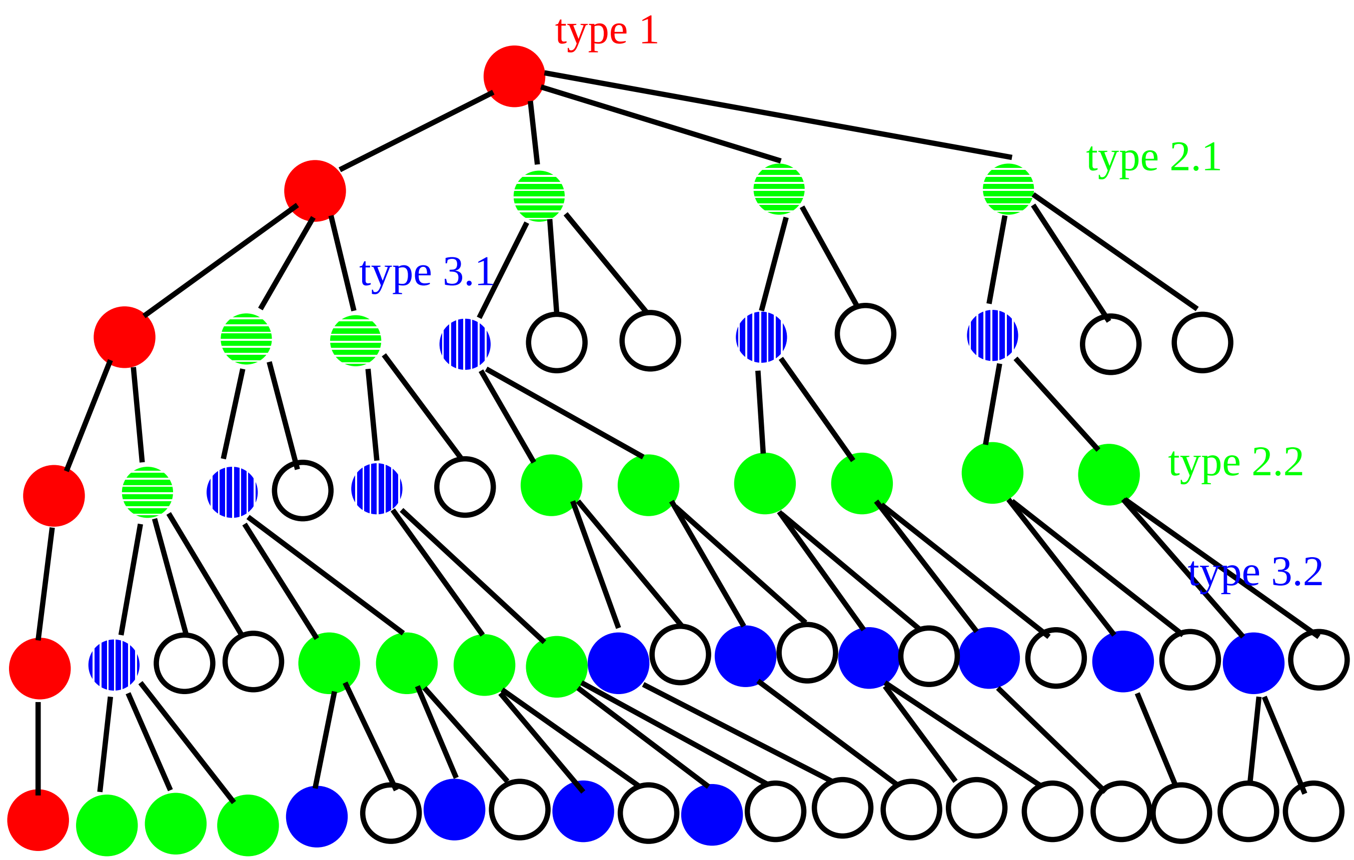


Illustration of all nodes

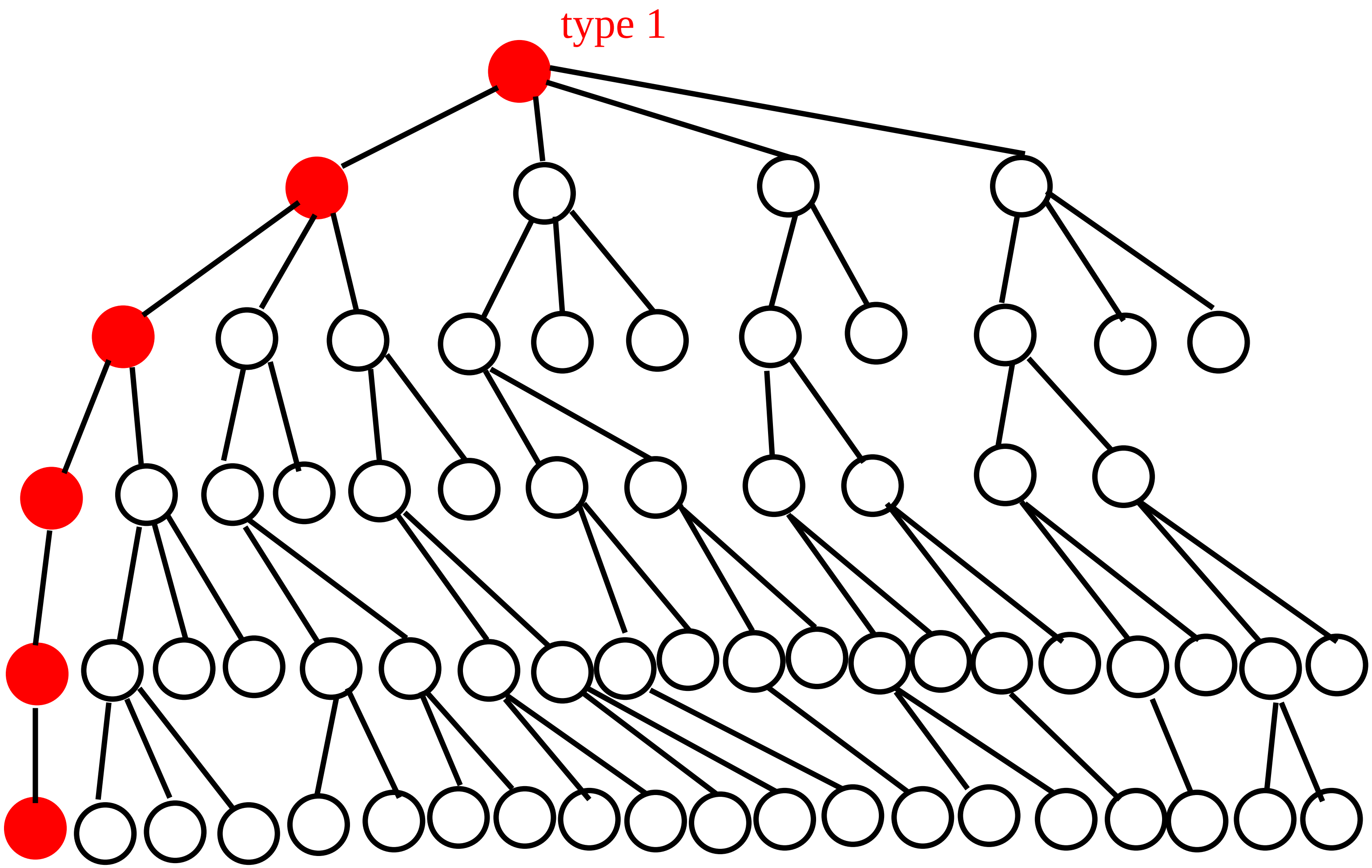


Illustration of all nodes

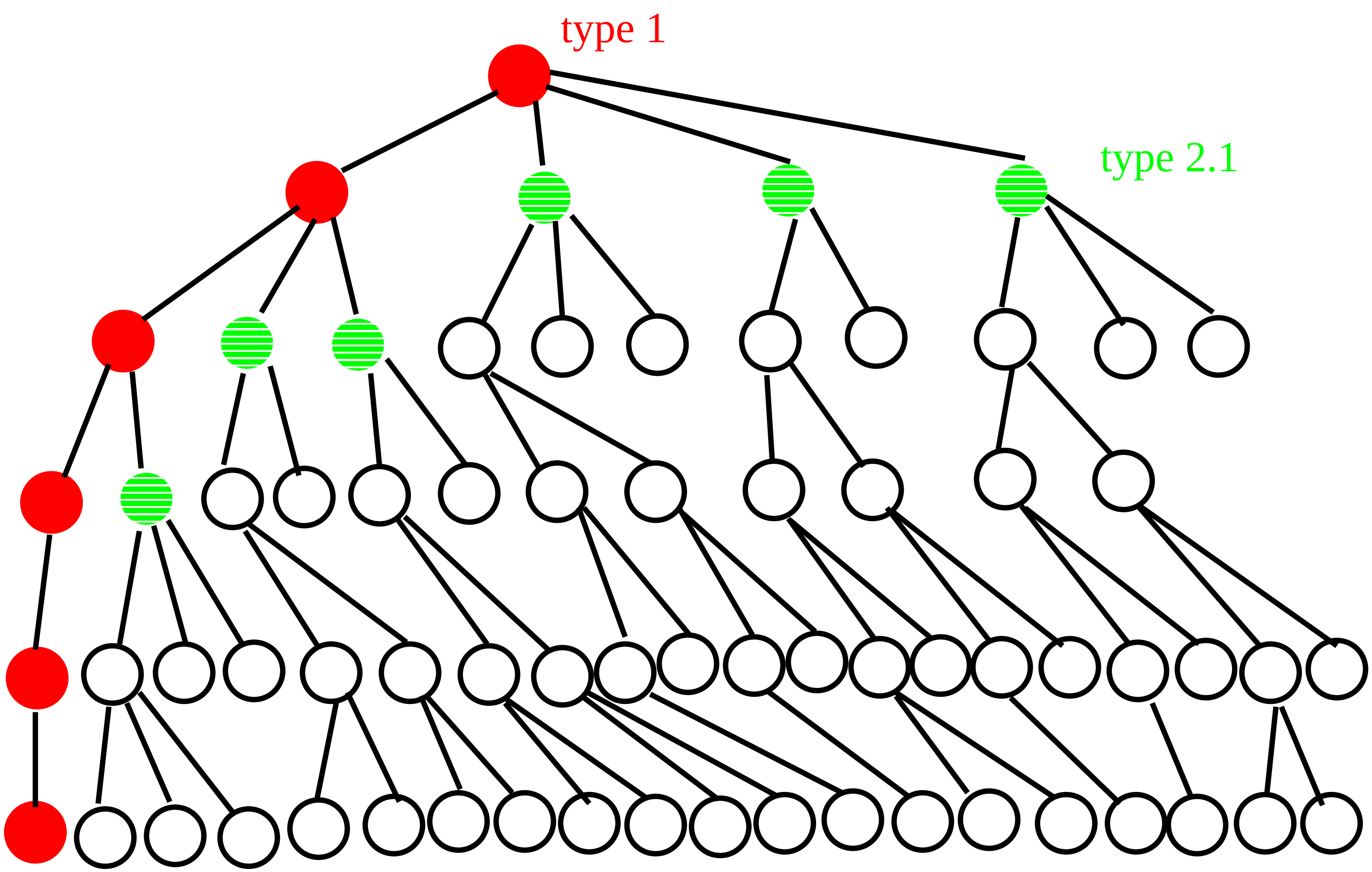


Illustration of all nodes

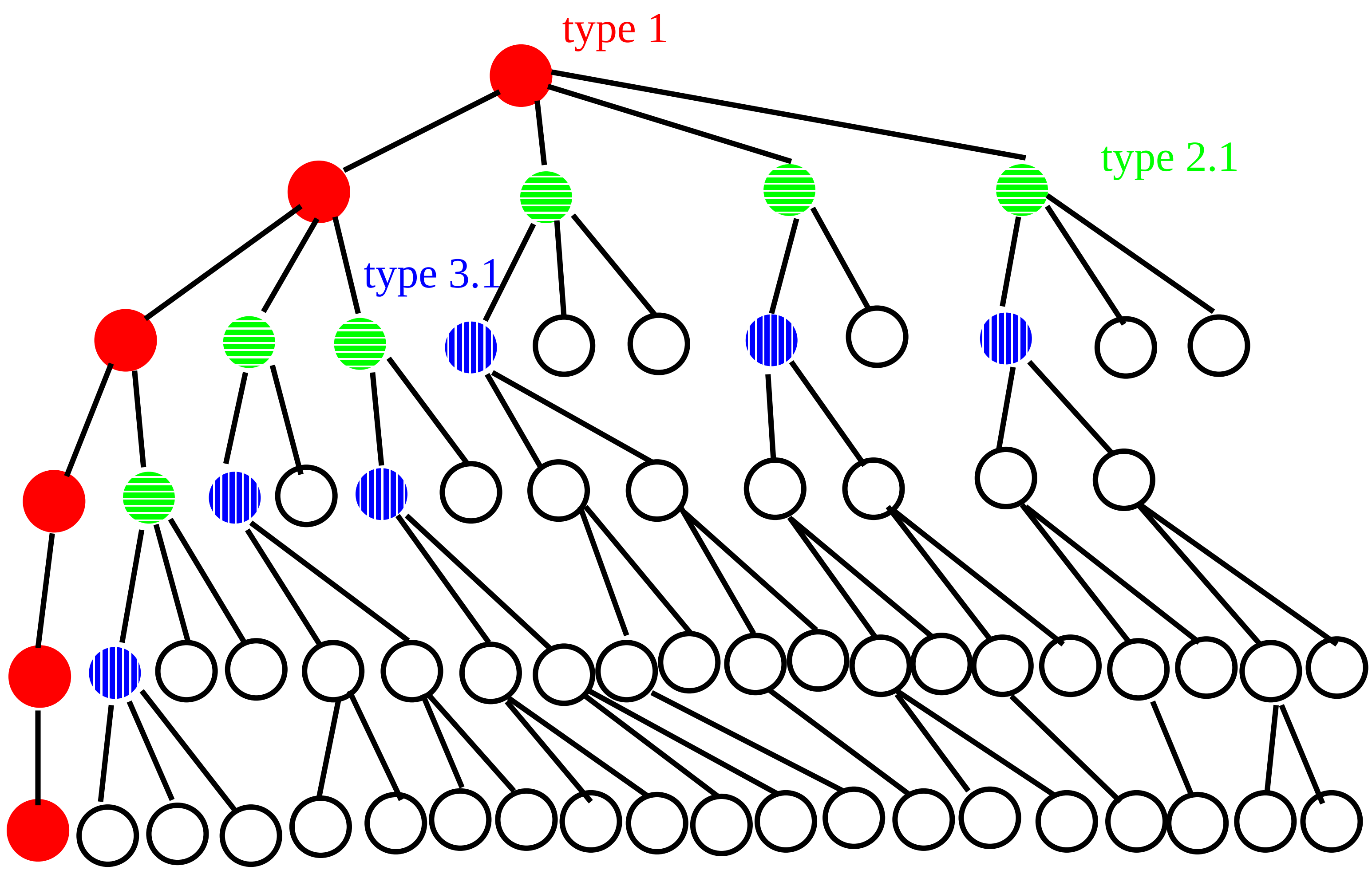


Illustration of all nodes

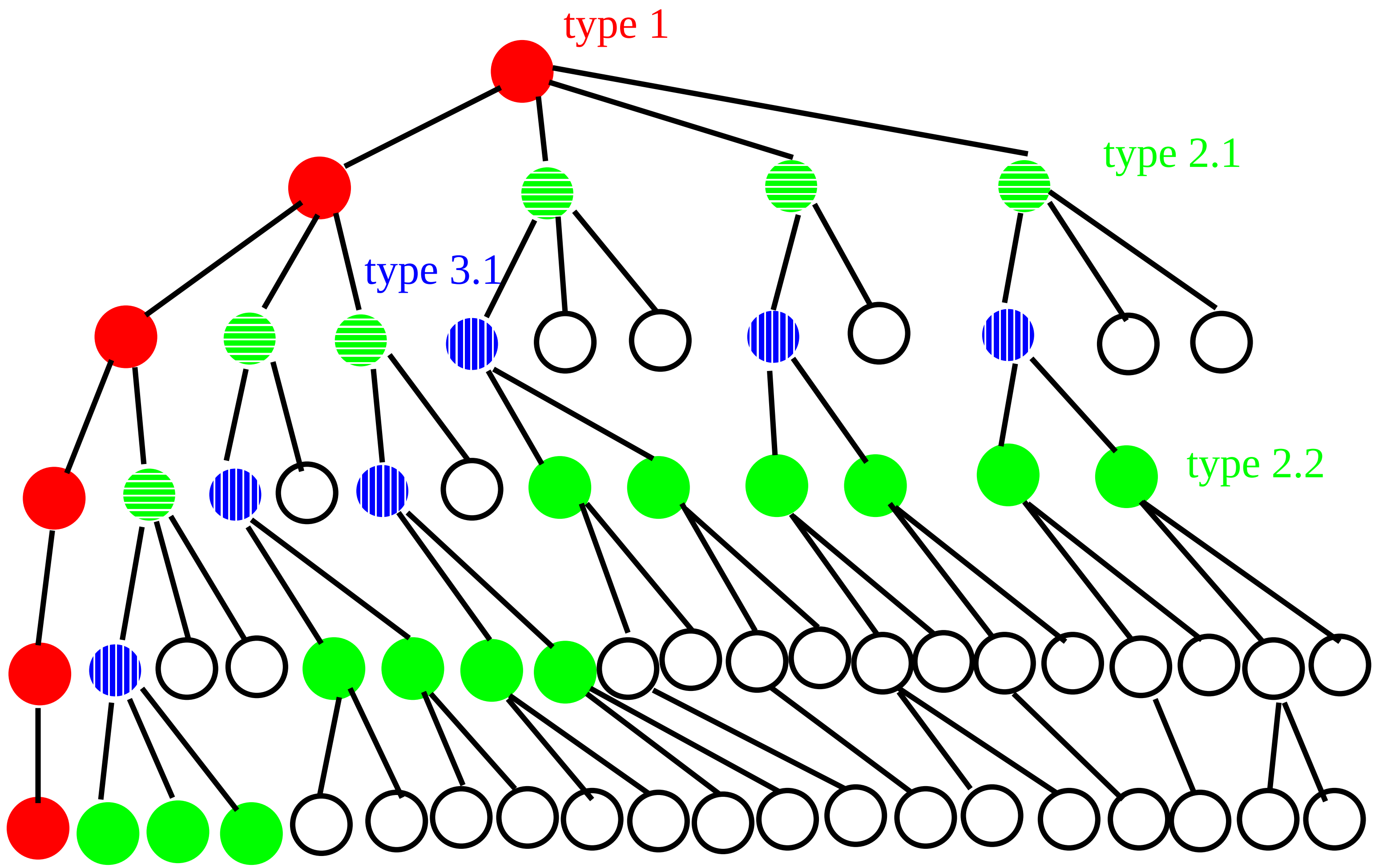


Illustration of all nodes

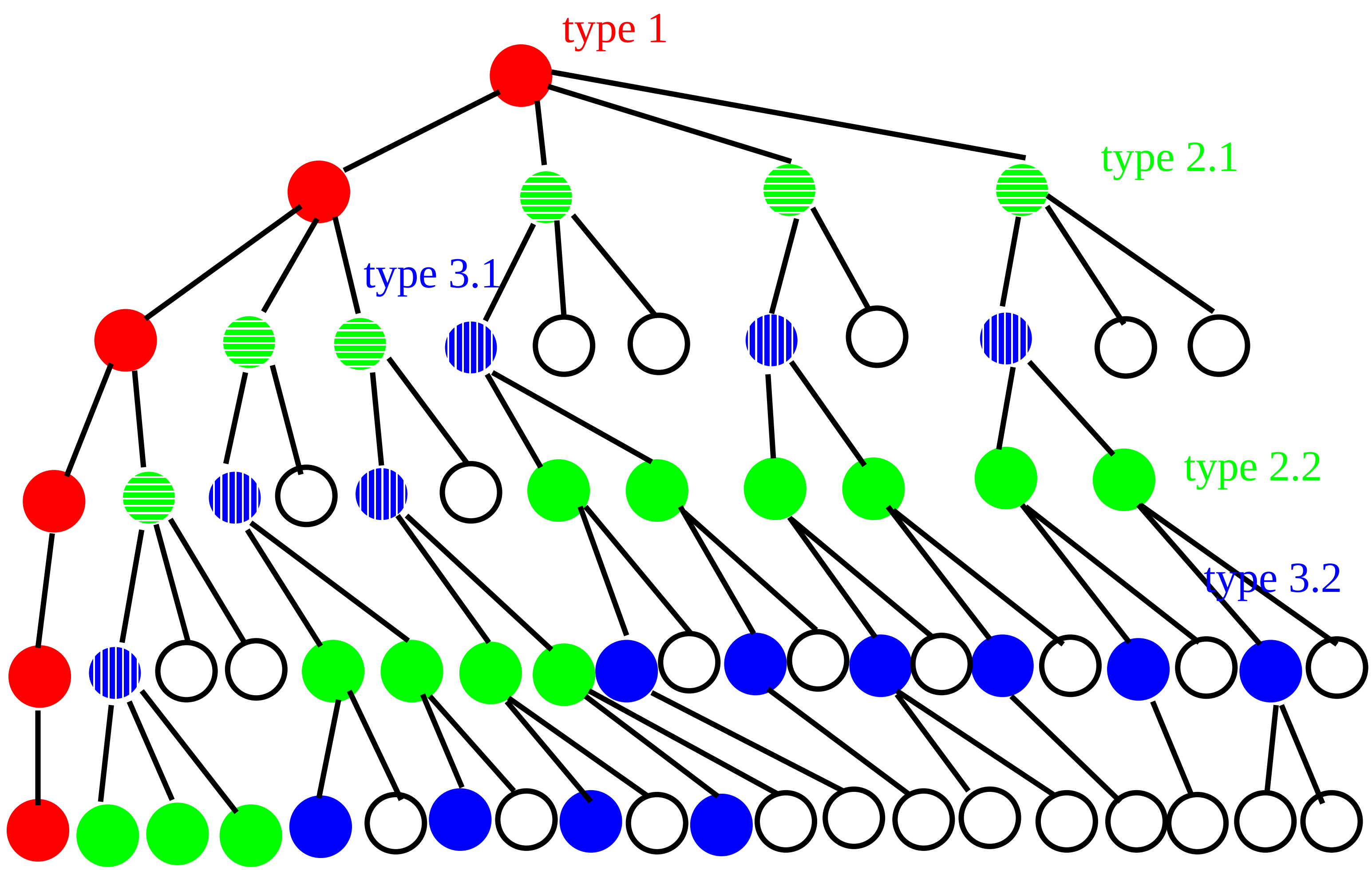


Illustration of all nodes

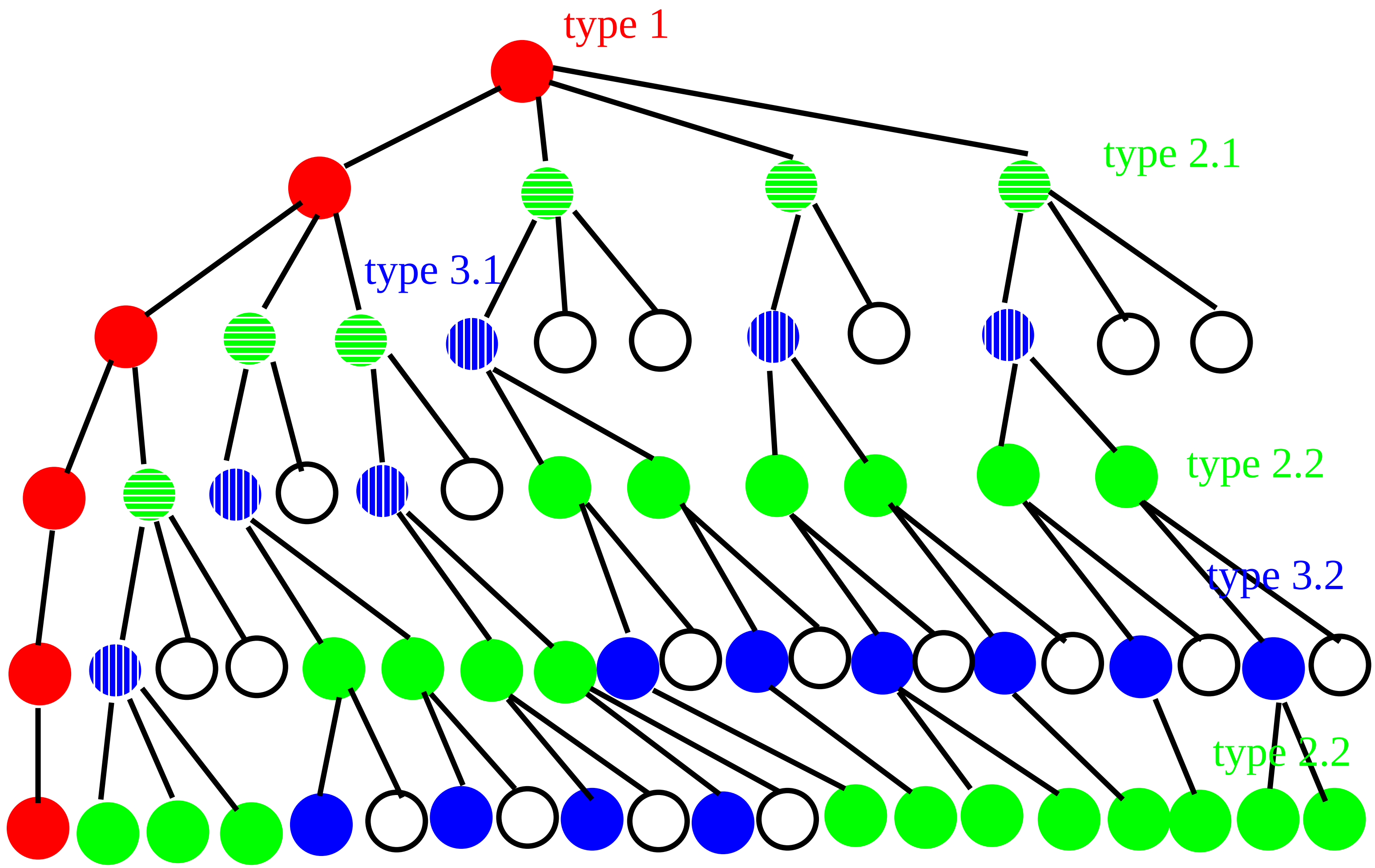
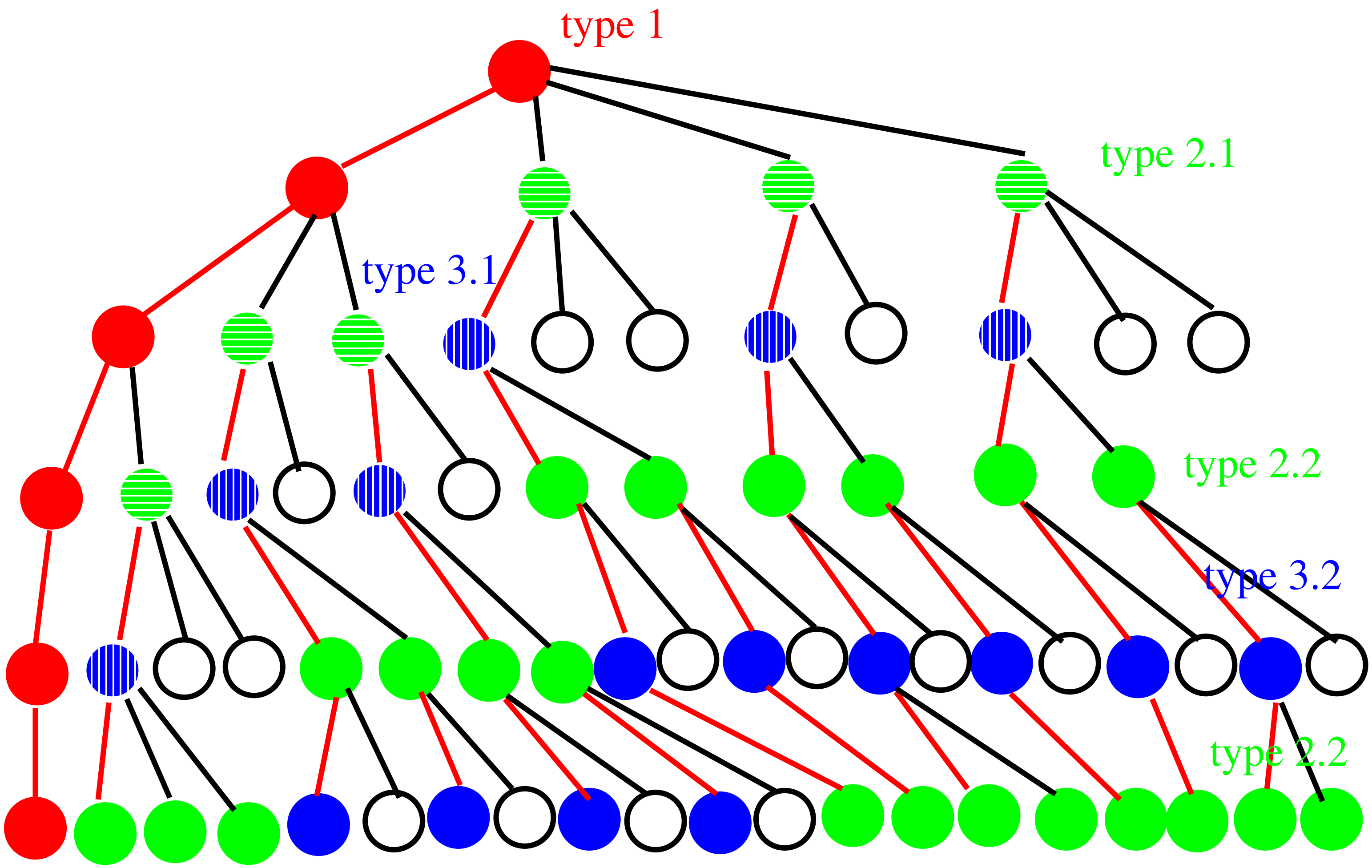


Illustration of all nodes



Theorem 1: Proof sketch

- **Properties (invariants)**
  - **A type 1 position p is examined by calling $F1(p, -\infty, \infty, depth)$**
    - ***p's first successor p_1 is of type 1***
    - $F(p) = -F(p_1) \neq \pm\infty$
    - ***p's other successors $p_2, \ldots, p_b$ are of type 2***
    - ***p_i, $i > 1$, are examined by calling $F1(p_i, -\infty, F(p_1), depth)$***
  - **A type 2 position p is examined by calling $F1(p, -\infty, beta, depth)$ where $-\infty < beta \leq F(p)$**
    - ***p's first successor p_1 is of type 3***
    - $F(p) = -F(p_1)$
    - ***p's other successors $p_2, \ldots, p_b$ are not examined***
  - **A type 3 position p is examined by calling $F1(p, alpha, \infty, depth)$ where $\infty > alpha \geq F(p)$**
    - ***p's successors $p_1, \ldots, p_b$ are of type 2***
    - ***they are examined by calling $F1(p_1, -\infty, -alpha, depth)$, $F1(p_2, -\infty, -\max\{m_1, alpha\}, depth), \ldots,$ $F1(p_i, -\infty, -\max\{m_{i-1}, alpha\}, depth)$ where $m_i = F1(p_i, -\infty, -\max\{m_{i-1}, alpha\}, depth)$***
- **Using an inductive argument to prove.**

Properties of Theorem 1

- **To cut off a subtree rooted at a node u entirely using alpha-beta based algorithms, at the very least, we need to know the values of**
  - **one of u's elder sibiling, and**
  - **one of v' elder sibiling where v is the parent of u.**
- **To know the value of a node rooted at a subtree, the subtree's left-most branch must be examined at the very least.**
- **Branches of a vertex that are examined**
  - **leftmost branch only**
    - ▷ ***type 2.1 to type 3.1***
    - ▷ ***type 2.2 to type 3.2***
  - **all branches**
    - ▷ ***type 1***
    - ▷ ***type 3.1***
    - ▷ ***type 3.2***

Analysis: best case

- **Corollary 1: Assume each position has exactly b successors**
  - **The number of positions examined by the alpha-beta procedure on level i is exactly**
    $$b^{\lceil i/2 \rceil} + b^{\lfloor i/2 \rfloor} - 1.$$
- **Proof:**
  - **There are $b^{\lfloor i/2 \rfloor}$ sequences of the form $a_1. \cdots .a_i$ with $1 \leq a_i \leq b$ for all i such that $a_i = 1$ for all odd values of i.**
  - **There are $b^{\lceil i/2 \rceil}$ sequences of the form $a_1. \cdots .a_i$ with $1 \leq a_i \leq b$ for all i such that $a_i = 1$ for all even values of i.**
  - **We subtract 1 for the sequence $1.1. \cdots .1.1$ which are counted twice.**
- **Total number of nodes visited is**

$$\sum_{i=0}^{\ell} b^{\lceil i/2 \rceil} + b^{\lfloor i/2 \rfloor} - 1.$$

Analysis: average case

- **Assumptions: Let a random game tree be generated in such a way that each position on level j has**
  - **a probability q_j of being nonterminal and**
  - **an average of b_j successors.**
- **Properties of the above random game tree**
  - **Expected number of positions on level ℓ is $b_0 \times b_1 \times \cdots \times b_{\ell-1}$**
  - **Expected number of positions on level ℓ examined by an alpha-beta procedure assumed the random game tree is perfectly ordered is**

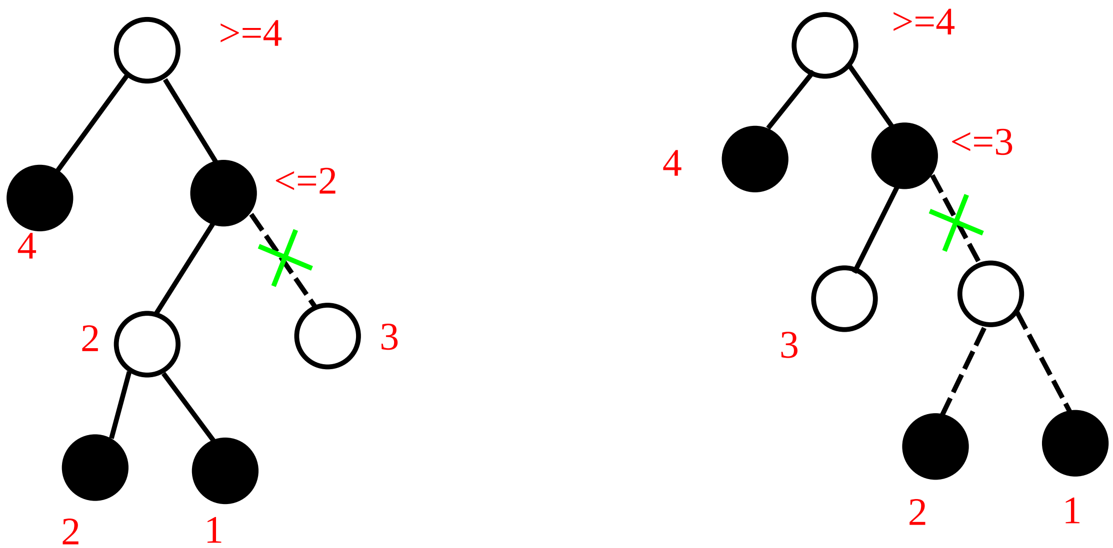
$$b_0 q_1 b_2 q_3 \cdots b_{\ell-2} q_{\ell-1} + q_0 b_1 q_2 b_3 \cdots q_{\ell-2} b_{\ell-1} - q_0 q_1 \cdots q_{\ell-1} \text{ if } \ell \text{ is even;}$$

$$b_0 q_1 b_2 q_3 \cdots q_{\ell-2} b_{\ell-1} + q_0 b_1 q_2 b_3 \cdots b_{\ell-2} q_{\ell-1} - q_0 q_1 \cdots q_{\ell-1} \text{ if } \ell \text{ is odd}$$

- **Proof sketch:**
  - **If x is the expected number of positions of a certain type on level j, then $x \times b_j$ is the expected number of successors of these positions, and $x \times q_j$ is the expected number of "numbered 1" successors.**
  - **The above numbers equal to those of Corollary 1 when $q_j = 1$ and $b_j = b$ for $0 \le j < \ell$.**

Perfect ordering is not always the best

- **Intuitively, we may "think" alpha-beta pruning would be most effective when a game tree is perfectly ordered.**
  - **That is, when the first successor of every position is the best possible move.**
  - **This is not always the case!**



- **Truly optimum order of game trees traversal is not obvious.**

When is a branch pruned?

- **Assume a node r has two children u and v with u being visited before v using some move ordering.**
  - **Further assume u produced a new bound $bound$.**
- **Assume node v has a child w.**
  - **If the value new returned from w can cause a range conflict with $bound$, then branches of v later than w are cut.**
- **This means as long as the "relative" ordering of u and v is good enough, then we can have a cut-off.**
  - **There is no need to have a perfect ordering to enable cut-off to happen.**

Theorem 2

- **Theorem 2: Alpha-beta pruning is optimum in the following sense:**
  - **Given any game tree and any algorithm which computes the value of the root position, there is a way to permute the tree**
    - *by reordering successor positions if necessary;*
  - **so that every terminal position examined by the alpha-beta method under this permutation is examined by the given algorithm.**
  - **Furthermore if the value of the root is not ∞ or $-\infty$, the alpha-beta procedure examines precisely the positions which are critical under this permutation.**

Variations of alpha-beta search

- **Initially, to search a tree with the root r by calling $F1(r,-\infty,+\infty,depth)$.**
  - **What does it mean to search a tree with the root r by calling $F1(r,alpha,beta,depth)$?**
    - ▷ ***To search the tree rooted at r requiring that the returned value to be within $alpha$ and $beta$.***
- **In an alpha-beta search with a pre-assigned window $[alpha, beta]$:**
  - **Failed-high means it returns a value that is larger than or equal to its upper bound $beta$.**
  - **Failed-low means it returns a value that is smaller than or equal to its lower bound $alpha$.**
- **Variations:**
  - **Brute force Nega-Max version: F**
    - ▷ ***Always finds the correct answer according to the Nega-Max formula.***
  - **Original alpha-beta cut (Nega-Max) version: $F1$**
  - **Fail hard alpha-beta cut (Nega-Max) version: $F2$**
  - **Fail soft alpha-beta cut (Nega-Max) version: $F3$**

Original version

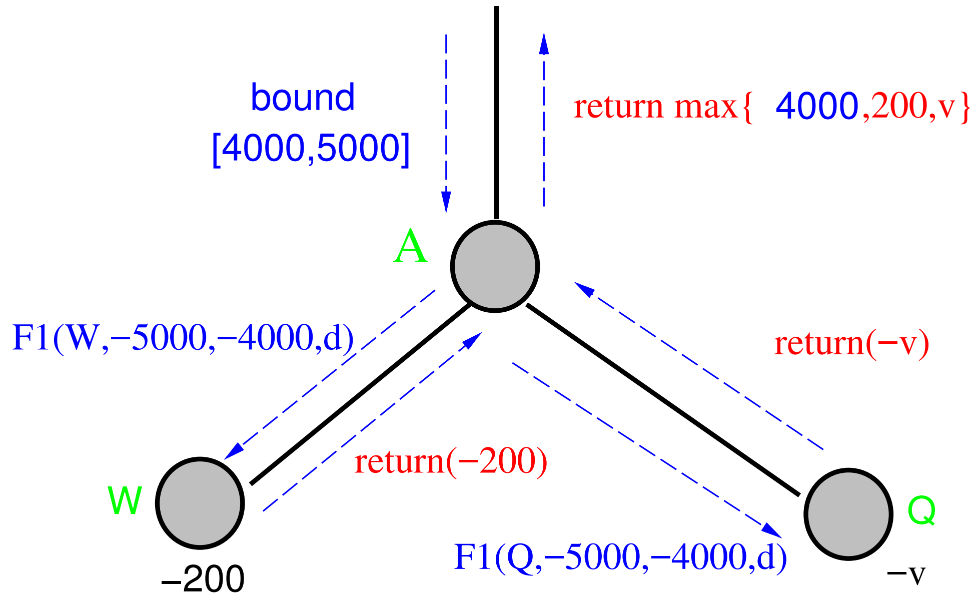
- **Requiring** $alpha \leq beta$
- **Algorithm** $F1$**(position** p**, value** $alpha$**, value** $beta$**, integer** $depth$**)**

  - **determine the successor positions** $p_1, \ldots, p_b$
  - **if** $b = 0$ **// a terminal node**
    or $depth = 0$ **// remaining depth to search**
    or time is running up // from timing control
    or some other constraints are met // add knowledge here
  - **then return** $h(p)$ **else**
  - **begin**
    - ▷ $m := alpha$ ***// hard initial value***
    - ▷ ***for*** $i := 1$ ***to*** b ***do***
    - ▷ ***begin***
    - ▷ $t := -F1(p_i, -beta, -m, depth - 1)$
    - ▷ ***if*** $t > m$ ***then*** $m := t$ ***// the returned value is "used"***
    - ▷ ***if*** $m \geq beta$ ***then return(***$beta$***) // cut off and return the hard bound***
    - ▷ ***end***
  - **end**
  - **return** m **// if nothing is over alpha, then alpha is returned**

Properties and comments

- **Properties:**
  - **Assumptions: (1)** $alpha < beta$ **and (2)** p **is not a leaf.**
  - $F1(p, alpha, beta, depth) = alpha$ **if** $F(p) \leq alpha$
  - $F1(p, alpha, beta, depth) = F(p)$ **if** $alpha < F(p) < beta$
  - $F1(p, alpha, beta, depth) = beta$ **if** $F(p) \geq beta$
  - $F1(p, -\infty, +\infty, depth) = F(p)$
- **Comments:**
  - $F1(p, alpha, beta, depth)$**: find the best possible value according to a nega-max formula for the position** p **with the constraints that**
    - ▷ ***If*** $F(p) \leq alpha$***, then*** $F1(p, alpha, beta, depth)$ ***returns with the value*** $alpha$ ***from a terminal position whose value is*** $\leq alpha$***.***
    - ▷ ***If*** $F(p) \geq beta$***, then*** $F1(p, alpha, beta, depth)$ ***returns the value*** $beta$ ***from a terminal position whose value is*** $\geq beta$***.***
  - **The meanings of** $alpha$ **and** $beta$ **during searching:**
    - ▷ ***For a max node: the current best value is at least*** $alpha$***.***
    - ▷ ***For a min node: the current best value is at most*** $beta$***.***
  - $F1$ **always finds a value that is within** $alpha$ **and** $beta$**.**
    - ▷ ***The bounds are hard, i.e., cannot be violated.***

Original version: Example



- **As long as the value of the leaf node W is less than the current $alpha$ value, the returned value of A will be $alpha$.**
- **If the value of the leaf node W is greater than the current $beta$ value, the returned value of A will be $beta$.**

Alpha-beta pruning algorithm: Fail hard

- **Algorithm $F2'$(position p, value $alpha$, value $beta$) // max node**
  - **determine the successor positions $p_1, \ldots, p_b$**
  - **if $b = 0$, then return $f(p)$ else begin**
    - ▷ $m := alpha$
    - ▷ ***for*** $i := 1$ ***to*** b ***do***
    - ▷ $\quad t := G2'(p_i, m, beta)$
    - ▷ $\quad$ ***if*** $t > m$ ***then*** $m := t$
    - ▷ $\quad$ ***if*** $m \geq beta$ ***then return(m) // beta cut off, return m***
  - **end; return m**
- **Algorithm $G2'$(position p, value $alpha$, value $beta$) // min node**
  - **determine the successor positions $p_1, \ldots, p_b$**
  - **if $b = 0$, then return $f(p)$ else begin**
    - ▷ $m := beta$
    - ▷ ***for*** $i := 1$ ***to*** b ***do***
    - ▷ $\quad t := F2'(p_i, alpha, m)$
    - ▷ $\quad$ ***if*** $t < m$ ***then*** $m := t$
    - ▷ $\quad$ ***if*** $m \leq alpha$ ***then return(m) // alpha cut off, return m***
  - **end; return m**

Alpha-beta pruning algorithm: Fail hard

- **Algorithm** $F2$(**position** p, **value** $alpha$, **value** $beta$, **integer** $depth$)
  - determine the successor positions $p_1, \ldots, p_b$
  - if $b = 0$ // a terminal node
    or $depth = 0$ // remaining depth to search
    or time is running up // from timing control
    or some other constraints are met // add knowledge here
  - then return $h(p)$ else
  - begin
    - ▷ $m := alpha$
    - ▷ ***for*** $i := 1$ ***to*** b ***do***
    - ▷ ***begin***
    - ▷ $t := -F2(p_i, -beta, -m, depth - 1)$
    - ▷ ***if*** $t > m$ ***then*** $m := t$
    - ▷ ***if*** $m \geq beta$ ***then*** ***return***(m) // ***cut off, return*** m ***that is*** $\geq beta$
    - ▷ ***end***
  - end
  - return m

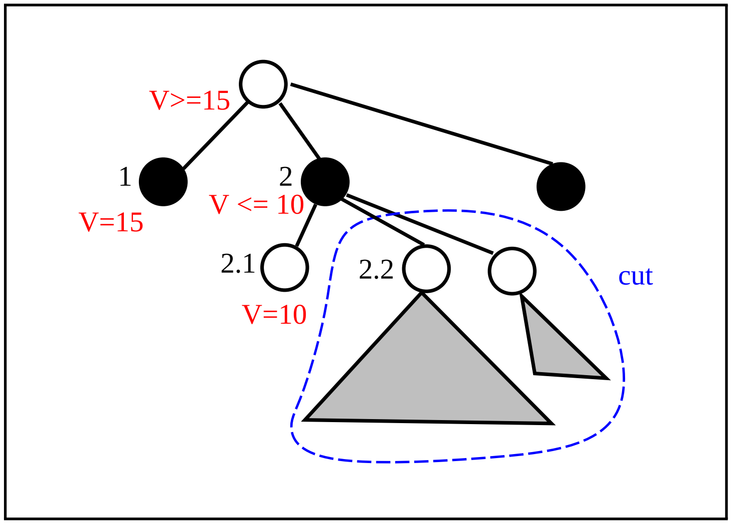
Properties and comments

- **Properties:**
  - **Assumptions: (1)** $alpha < beta$ **and (2)** p **is not a leaf.**
  - $F2(p, alpha, beta) = alpha$ **if** $F(p) \leq alpha$
  - $F2(p, alpha, beta) = F(p)$ **if** $alpha < F(p) < beta$
  - $F2(p, alpha, beta) \geq beta$ **and** $F(p) \geq F2(p, alpha, beta)$ **if** $F(p) \geq beta$
  - $F2(p, -\infty, +\infty) = F(p)$
- **Comments:**
  - $F2(p, alpha, beta)$**: find the best possible value according to a nega-max formula for the position** p **with the constraints that**
    - ***If*** $F(p) \leq alpha$***, then*** $F2(p, alpha, beta)$ ***returns with the value*** $alpha$ ***from a terminal position whose value is*** $\leq alpha$***.***
    - ***If*** $F(p) \geq beta$***, then*** $F2(p, alpha, beta)$ ***returns a value*** $\geq beta$ ***from a terminal position whose value is*** $\geq beta$***.***
  - **An intermediate version.**
    - ***The lower bound is hard, cannot be violated.***
    - ***Easier to find the branch where the returned value is coming from.***
    - ***Always return something better than expected, but never something worse!!***
  - **For historical reason [Fishburn 1983][Knuth & Moore 1975], this is called fail hard.**

Example

Initial call: $F2'(\text{root},-\infty,\infty)$

- $m = -\infty$
- **call** $G2'$**(node 1,**$-\infty$**,**∞**)**
  - ▷ ***it is a terminal node***
  - ▷ ***return value*** 15
- $t = 15$**;**
  - ▷ ***since*** $t > m$***,*** m ***is now*** 15
- **call** $G2'$**(node 2,**15**,**∞**)**
  - ▷ ***call*** $F2'$***(node 2.1,***15***,***∞***)***
  - ▷ ***it is a terminal node; return*** 10
  - ▷ $t = 10$***; since*** $t < \infty$***,*** m ***is now*** 10
  - ▷ $alpha$ ***is*** 15***,*** m ***is*** 10***, so we have an alpha cut off,***
  - ▷ ***no need to call*** $F2'$***(node 2.2,15,10)***
  - ▷ ***return 10***
  - ▷ $\cdots$



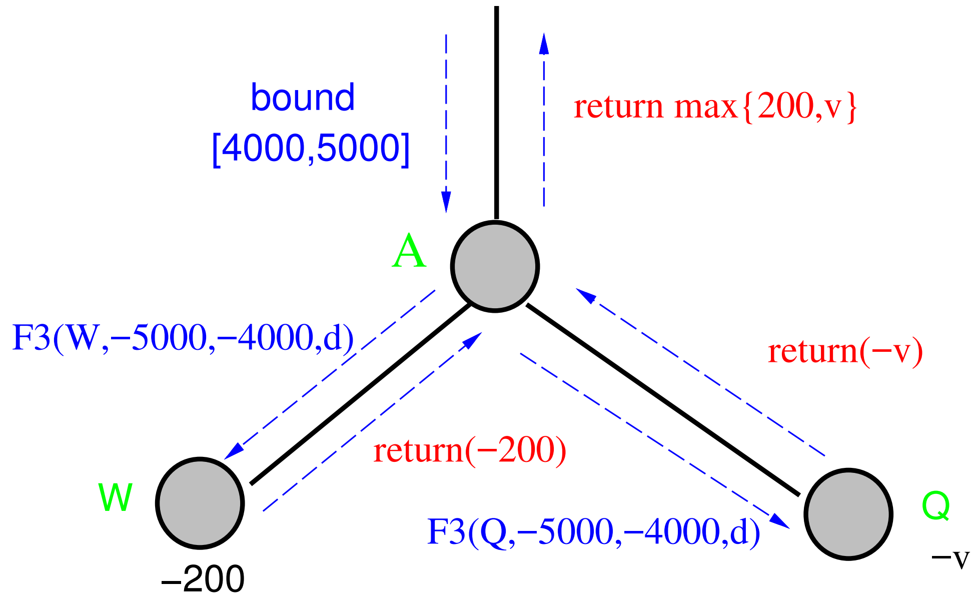
Fail soft version

- **Algorithm** $F3$**(position** p**, value** $alpha$**, value** $beta$**, integer** $depth$**)**
  - **determine the successor positions** $p_1, \ldots, p_b$
  - **if** $b = 0$ **// a terminal node**
    or $depth = 0$ **// remaining depth to search**
    or time is running up // from timing control
    or some other constraints are met // add knowledge here
  - **then return** $h(p)$ **else**
  - **begin**
    - ▷ $m := -\infty$ ***// soft initial value***
    - ▷ ***for*** $i := 1$ ***to*** b ***do***
    - ▷ ***begin***
    - ▷ $t := -F3(p_i, -beta, -\max\{m, alpha\}, depth - 1)$
    - ▷ ***if*** $t > m$ ***then*** $m := t$ ***// the returned value is "used"***
    - ▷ ***if*** $m \geq beta$ ***then return(***m***) // cut off***
    - ▷ ***end***
  - **end**
  - **return** m

Properties and comments

- **Properties:**
  - **Assumptions (1)** $alpha < beta$ **and (2)** p **is not a leaf**
  - $F3(p, alpha, beta, depth) \leq alpha$ **and** $F(p) \leq F3(p, alpha, beta, depth)$ **if** $F(p) \leq alpha$
  - $F3(p, alpha, beta, depth) = F(p)$ **if** $alpha < F(p) < beta$
  - $F3(p, alpha, beta, depth) \geq beta$ **and** $F(p) \geq F3(p, alpha, beta, depth)$ **if** $F(p) \geq beta$
  - $F3(p, -\infty, +\infty, depth) = F(p)$
- $F3$ **finds a "better" value when the value is out of the search window.**
  - **Better means a tighter bound.**
    - ▷ *The bounds are soft, i.e., can be violated.*
  - **When it is failed-high,** $F3$ **normally returns a value that is higher than that of** $F1$ **or** $F2$**.**
    - ▷ *Never higher than that of F!*
  - **When it is failed-low,** $F3$ **normally returns a value that is lower than that of** $F1$ **or** $F2$**.**
    - ▷ *Never lower than that of F!*

Fail soft version: Example

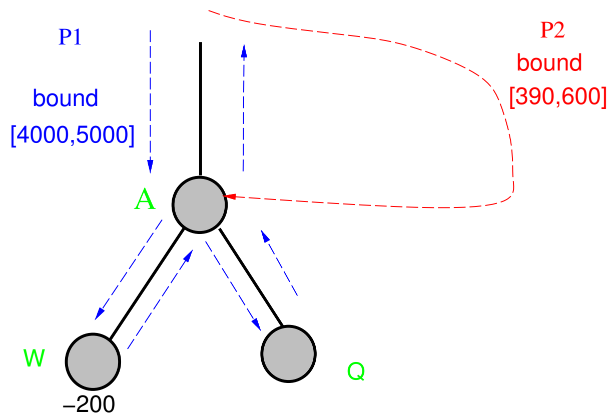


- **Let the value of the leaf node W be u.**
- **If $u < alpha$, then the returned value of A will be at least u.**

Comparisons between $F2$ and $F3$

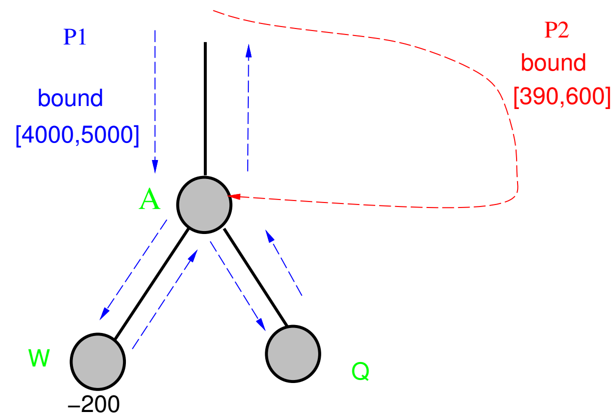
- Both versions find the corrected value v if v is within the window $[alpha, beta]$.
- Both versions scan the same set of nodes during searching.
  - ▷ *If the returned value of a subtree is decided by a cut, then $F2$ and $F3$ return the same value.*
- $F3$ provides more information when the true value is out of the pre-assigned search window.
  - Can provide a feeling on how bad or good the game tree is.
  - Use this "better" value to guide searching later on.
- $F3$ saves about 7% of time than that of $F2$ when a transposition table is used to save and re-use searched results [Fishburn 1983].
  - A transposition table is a data structure to record the results of previous searched results.
  - The entries of a transposition table can be efficiently accessed, i.e., read and write, during searching.
  - Need an efficient addressing scheme, e.g., hash, to translate between a position and its address.

$F2$ and $F3$: Example (1/2)



- **Assume the node A can be reached from the starting position using path P_1 and path P_2.**
  - **If W is visited first along P_1 with a bound of $[4000, 5000]$, and returns a value of 200, then**
    - ▷ ***the returned value of W, 200, is stored into the transposition table.***
  - **If A is visited again along P_2 with a bound of $[390, 600]$, then a better value of previously stored value of W helps to decide whether the subtree rooted at W needs to be searched again.**

$F2$ and $F3$: Example (2/2)



- **Fail soft version has a chance to record a better value to be used later when this position is revisited.**
  - **If A is visited again along P_2 with a bound of $[390, 600]$, then**
    - ▷ *it does not need to be searched again, since the previous stored value of W is -200.*
  - **However, if the value of W is 450, then it needs to be searched again.**
- **Fail hard version does not store the returned value of W after its first visit since this value is less than $alpha$.**

Comments

- For historical reason, comparisons are made between $F2$ and $F3$, while we should compare $F1$ and $F3$.
  - To me, $F1$ fails really hard. $F2$ is only an intermediate version!
- What move ordering is good?
  - It may not be good to search the best possible move first.
  - It may be better to cut off a branch with more nodes first.
- How about the case when the tree is not uniform?
- What is the effect of using iterative-deepening alpha-beta cut off?
- How about the case for searching a game graph instead of a game tree?
  - Can some nodes be visited more than once?

References and further readings

* D. E. Knuth and R. W. Moore. An analysis of alpha-beta pruning. *Artificial Intelligence*, 6:293–326, 1975.
* John P. Fishburn. Another optimization of alpha-beta search. *SIGART Bull.*, (84):37–38, 1983.
- J. Pearl. The solution for the branching factor of the alpha-beta pruning algorithm and its optimality. *Communications of ACM*, 25(8):559–564, 1982.