Undirected Vertex-Connectivity Structure and Smallest Four-Vertex-Connectivity Augmentation

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Abstract

In this paper, we study properties for the structure of an undirected graph that is not 4-vertex-connected. We also study the evolution of this structure when an edge is added to optimally increase the vertex-connectivity of the underlying graph. Several properties reported here can be extended to the case of a graph that is not k-vertex-connected, for an arbitrary k.

Using properties obtained here, we solve the problem of finding a smallest set of edges whose addition 4-vertex-connects an undirected graph. This is a fundamental problem in graph theory and has applications in network reliability and in statistical data security. We give an $O(n \cdot \log n + m)$-time algorithm for finding a set of edges with the smallest cardinality whose addition 4-vertex-connects an undirected graph, where n and m are the number of vertices and edges in the input graph, respectively. This is the first polynomial time algorithm for this problem when the input graph is not 3-vertex-connected. We also show a formula to compute this smallest number in $O(n \cdot \alpha(n, n) + m)$ time, where α is the inverse of the Ackermann function. This is also the first polynomial time algorithm for computing this number when the input graph is not 3-vertex-connected. Our algorithm can also be used to find a smallest k-vertex-connectivity augmentation, for any $k \leq 3$.

1 Introduction

In studying the vertex-connectivity of an undirected graph, it is desirable to know the set of separating sets and the set of subsets of vertices that have higher connectivity than the original graph. It is also desirable to know the relations between the two sets. A clear, compact, and systematic description of the above information is the *structure* of the graph with respect to its vertex-connectivity. Knowing the structure of a graph can lead to the solution of important graph-theoretical problems such as the augmentation problem that is studied here (see the survey chapter in [Hsu93]) and dynamic graph algorithms [LP91].

The structure of an undirected graph that is not biconnected (i.e., 2-vertex-connected) is well-known [Har69] and is represented as a 2-block graph. The structure of a biconnected graph that is not triconnected (i.e., 3-vertex-connected) is also well-known and is represented as a 3-block graph [HT73, Tut66]. The 3-block graph is also extended for a non-biconnected graph [DBT90, HR91]. Recently, the structure of a graph that is not four-connected (i.e., 4-vertex-connected) is studied in [Hsu92, KR91, KTDBC91] with the emphasis on triconnected graphs. Several properties of a $(k-1)$-vertex-connected graph that is not k-vertex-connected are also studied in [CBKT93, Mat72, Mat78]. In this paper, we study the structure of an undirected graph that is not four-connected detailing on the parts we need to solve the smallest four-connectivity augmentation problem. We also study the evolution of this structure when an edge is added to optimally increase the vertex-connectivity of the underlying graph. Several properties given here can be extended to the case of a graph that is not k-vertex-connected, for an arbitrary k. Though these studies, we solve the *smallest augmentation* problem for making an undirected graph four-connected.

Algorithmic Results

The problem of augmenting a graph to reach a given connectivity requirement by adding edges has important applications in network reliability [FC70, JG86, SWK69] and in statistical database security [Cox75, Gus89, KG93]. One version of the augmentation problem is to satisfy the given requirement by adding as few edges as possible. We refer to this problem as the *smallest augmentation* problem.

In solving the smallest vertex-connectivity augmentation problem, a framework of finding a smallest augmentation for the case of increasing the vertex-connectivity by one is first reported in [ET76] for the case of reaching biconnectivity. This framework has been extended for reaching triconnectivity [HR91, Jor93b], and for four-connectivity [Hsu92]. However, it is difficult to see a general framework for the case of increasing the vertex-connectivity by more than one from the results in [HR91, WN93]. In [HR91], a counter example is given to show that we cannot optimally raise the vertex-connectivity of a graph to three by first optimally raising the vertex-connectivity to two and then using the special algorithm to increase the vertex-connectivity by one. (This approach is used to optimally raise the edge-connectivity of a graph [Fra92, Gab91, NGM90].) The above counter example can be extended to rule out the chance of solving our problem (for raising the vertex-connectivity to four) by combining the result in [HR91] (for raising the vertex-connectivity to three) and the result in [Hsu92] (for raising the vertex-connectivity from three to four).

Using properties obtained in this paper, we study the smallest four-connectivity augmentation problem. We are unaware of any polynomial time algorithm for this problem when the input graph is not triconnected. In this paper, we give an $O(n \cdot \log n + m)$-time algorithm to solve this problem, where n and m are the number of vertices and edges in the input graph, respectively. We also show a formula to compute this smallest number in $O(n \cdot \alpha(n, n) + m)$ time, where α is the inverse of the Ackermann function. This is the first polynomial time algorithm to compute this number exactly. Our algorithm can also be used to find a smallest k-vertex-connectivity augmentation, for any $k \leq 3$.

In developing our algorithm for increasing the vertex-connectivity of a (possibly) disconnected graph to four, we establish theorems that might be useful in answering questions arising from solving the more difficult problem of raising the vertex-connectivity by an arbitrary value. Let $\mathcal{A}_{x,y}$ be an algorithm that optimally raises the vertex-connectivity of an x-vertex-connected graph to y. We found although that we cannot derive $\mathcal{A}_{0,4}$ by sequentially first applying $\mathcal{A}_{0,3}$ and then applying $\mathcal{A}_{3,4}$, we can use the information available in computing $\mathcal{A}_{0,3}$ and $\mathcal{A}_{3,4}$ together with additional information we found to derive $\mathcal{A}_{0,4}$. We also show that the same approach can be used to construct $\mathcal{A}_{0,3}$ from $\mathcal{A}_{0,2}$ and $\mathcal{A}_{2,3}$.

The algorithmic notation used is pseudo-Pascal and is similar to the notation of Tarjan [Tar83] and Ramachandran [Ram93]. We enclose comments between '$\{*$' and '$*\}$'.

The organization of this paper is as follows. In Section 2, we survey related work. In Section 3, we give definitions used in this paper. We then describe properties of blocks in Section 4, properties of separating sets in Section 5, and properties of wheels in Section 3.3. We give methods for maintaining and updating blocks, separating sets, and wheels when an edge is added during the process of finding a smallest augmentation. In Section 8, we give our algorithms for computing the smallest four-connectivity augmentation number and for finding such a smallest four-connectivity augmentation. Finally, we give concluding remarks in Section 9.

2 Related Work

We give a brief summary of related work in this section. More details can be found in the survey chapter in [Hsu93].

2.1 Vertex-Connectivity Augmentation

The following results are known for solving the smallest augmentation problem on an undirected graph to satisfy a given vertex-connectivity requirement.

Eswaran and Tarjan [ET76] (and Plesník [Ple76], independently) gave a lower bound for the smallest number of edges needed to biconnect an undirected graph and proved that the lower bound can always be achieved. Rosenthal and Goldner [RG77] developed a linear-time sequential algorithm for finding a smallest biconnectivity augmentation; however, the algorithm in [RG77] contains an error. Hsu and Ramachandran [HR93] gave a corrected linear-time sequential algorithm. An $O(\log^2 n)$-time parallel algorithm on an EREW PRAM using a linear number of processors for this problem was also given in Hsu and Ramachandran [HR93].

Fernández-Baca and Williams [FBW89] considered the smallest augmentation problem for reaching biconnectivity on hierarchically defined graphs. This version of the augmentation problem has applications in VLSI circuit design. They obtained a polynomial time algorithm for the above problem.

Watanabe and Nakamura [WN93] gave an $O(n \cdot (n+m)^2)$-time sequential algorithm for finding a smallest augmentation to triconnect a graph with n vertices and m edges. Hsu and Ramachandran [HR91] gave a linear-time algorithm for this problem. (Independently, Jordán [Jor93b] gave a different linear-time algorithm for the special case of optimally triconnecting a biconnected graph.) Hsu [Hsu92] also gave an almost linear-time algorithm for four-connecting a triconnected graph by adding as few edges as possible.

There is no polynomial time algorithm known for finding a smallest augmentation to k-vertex-connect an undirected graph, for $k > 4$. Although no polynomial time solution is known for this problem, Jordán [Jor93b] gave an approximation algorithm for undirected graphs that uses no more than $k-3$ edges to k-vertex-connect a $(k-1)$-vertex-connected graph. There are also results known for augmenting planar graphs and outerplanar graphs [Kan93b].

The above results are for augmenting undirected graphs. For directed graph augmentation, Masuzawa, Hagihara, and Tokura [MHT87] studied this problem when the input graph is a directed oriented tree. Their algorithm runs in $O(\lambda \cdot n)$ time where λ is the vertex-connectivity of the resulting graph. Jordán [Jor93a] gave a polynomial time approximation algorithm that uses no more than k extra edges for augmenting a $(k-1)$-vertex-connected directed graph to achieve k-vertex-connectivity. Very recently, Frank and Jordán [FJ93] gave a polynomial time algorithm to solve the smallest vertex-connectivity augmentation problem on directed graphs exactly. Their algorithm increases the vertex-connectivity of a directed graph by any given δ optimally.

2.2 Edge-Connectivity Augmentations

For the problem of finding a smallest augmentation for a graph to reach a given edge-connectivity, several polynomial time algorithms and efficient parallel algorithms on outer-planar graphs, hierarchically defined graphs, undirected graphs, directed graphs and mixed graphs are known. These results can be found in [Ben94, CS89, ET76, FBW89, Fra92, Gab91, Gus87, Hsu93, KU86, Kan93b, NGM90, Sor88, TW94, UKW88, Wat87, WN87, WY93].

2.3 Augmenting a Weighted Graph

Another version of the problem is to augment a graph, with a weight assigned to each edge, to meet a connectivity requirement using a set of edges with a minimum total cost. The decision version of several related problems have been proved to be **NP**-hard. These results can be found in [ET76, Fra92, FJ81, KT92, WHN90, WN93].

3 Definitions

We use the following notations on graphs. In this paper, G is an undirected graph with the set of vertices V and the set of edges E and is also denoted as $G = (V, E)$. The graph G is simple, i.e., one without multiple edges between a pair of vertices in G and without self-loops. If u and v are two vertices in V, then (u, v) represents an edge between u and v. Given a connected component H in G, V_H is the set of vertices in H. Let U be a set of vertices in G. The graph $G - U$ is the induced subgraph of G on $V \setminus U$. Let E' be a subset of edges in E. The graph $G - (E' \cup U)$ is the resulting graph obtained from $G - U$ after removing edges in E'. Let E'' be a set of edges such that the two endpoints of each edge are in V. The graph $G \cup E''$ is the graph with the set of vertices V and the set of edges $E \cup E''$.

We then give definitions used in this paper.

3.1 Vertex-Connectivity

A graph is disconnect if there is no path between two distinct vertices. The graph G with at least $k + 1$ vertices is *k-vertex-connected*, $k \geq 2$, if and only if G is a complete graph with $k+1$ vertices or the removal of any set of vertices with cardinality less than k does not disconnect G [Bol79]. The *vertex-connectivity* of G is k if G is k-vertex-connected, but not $(k + 1)$-vertex-connected.

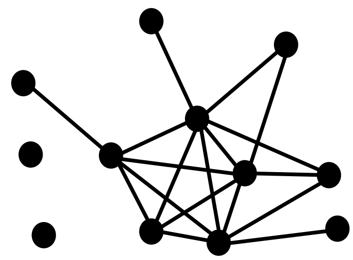

Figure 1: Illustrating separating sets in a graph. Vertices 1, 2, and 4 are cutpoints, while vertex 8 is not. The set $\{2, 3\}$ is a separating pair, while $\{1, 2\}$ is not. The set $\{2, 3, 4\}$ is a separating triplet, while $\{1, 2, 3\}$ is not.

Another characterization of k-vertex-connected graphs is due to Menger [Eve79, Men27]. Given a path[1] P, the *internal vertices* of P are vertices in P that are not its two endpoints. Two paths P_1 and P_2 are *internally vertex-disjoint* is there is no vertex that is both an internal vertex of P_1 and P_2. A set of paths W are *internally vertex-disjoint* if either $|W| = 1$ or every two distinct paths in W are internally vertex-disjoint. Two vertices are *k-vertex-connected*, if there are at least k internally vertex-disjoint paths between them. A set of vertices U is *k-vertex-connected*, if either (1) $|U| = 1$ and the degree of the vertex in U is at least k or (2) $|U| > 1$ and every two distinct vertices are k-vertex-connected. A graph $G = (V, E)$ is k-vertex-connected, $k \geq 2$, if G contains more than k vertices and V is k-vertex-connected. The above two definitions for k-vertex-connectivity are equivalent.

3.2 Separating Set

Given a subset of vertices $\mathcal{S}$ in G, $\mathcal{S}$ *separates* two vertices u and v in G (or *separates* u from v) if $u \notin \mathcal{S}$, $v \notin \mathcal{S}$, and u and v are connected in G, but are disconnected in $G - \mathcal{S}$. Let $\text{COM}(G)$ be the number of connected components in G. A set of vertices $\mathcal{S}$ in G is a *separating set* if there are two distinct vertices $u \notin \mathcal{S}$ and $v \notin \mathcal{S}$ in G such that (1) there are $|\mathcal{S}|$ internally vertex-disjoint paths between every two distinct vertices in $\mathcal{S} \cup \{u, v\}$, and (2) u and v are separated in $G - \mathcal{S}$.

If the cardinality of $\mathcal{S}$ is k, then $\mathcal{S}$ is a separating k-set. For the case of $k = 1$, the vertex in $\mathcal{S}$ is a *cutpoint*. If $k = 2$, it is a *separating pair*. If $k = 3$, it is a *separating triplet*. An example is illustrated in Figure 1. If the vertex-connectivity of G is k, then the cardinality of every separating set in G is at least k. Note also that the vertex-connectivity of a complete graph with $k+1$ vertices is k, but this graph contains no separating set. If the

[1] In this paper, any path is simple, i.e., one without passing a vertex twice, unless stated otherwise.

vertex-connectivity of G is k, G might contain separating sets with cardinality more than k. It is also possible that a separating set might properly contain a separating set. A set of separating sets with the same cardinality can have non-trivial intersections. This structure is a *wheel* and is elaborated in Section 3.3.

The identification of all separating sets with cardinality less than k is crucial in augmenting a graph to reach k-vertex-connectivity, since a k-vertex-connected graph contains no such separating set.

Regular Separating Set

Two separating ℓ-sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are *neighbors* in G if (1) the number of ℓ-components in $G - \mathcal{S}_1$ is equal to the number of ℓ-components in $G - \mathcal{S}_2$, and (2) $\mathcal{S}_1$ separates two ℓ-vertex-connected vertices u and v if and only if $\mathcal{S}_2$ separates u and v.

An *ℓ-hybrid-set* for G is $\mathcal{H} = \{w_i \mid 1 \leq i \leq \ell\}$, where each w_i is either an edge or a vertex. A *realization* for an ℓ-hybrid-set $\mathcal{H}$ is $\{r_i \mid i \leq \ell\}$ where $r_i = w_i$, $1 \leq i \leq \ell$, if w_i is a vertex and r_i is an endpoint of w_i if w_i is an edge. An ℓ-hybrid-set is a *separating ℓ-hybrid-set* if its every realization is a separating ℓ-set and every two distinct realizations are neighbors. Given a separating ℓ-hybrid-set $\mathcal{H}$ with at least one edge in $\mathcal{H}$, we can find exactly two realizations $\mathcal{S}_1$ and $\mathcal{S}_2$ with $\mathcal{S}_1 \cap \mathcal{S}_2$ equals to the set of vertices in $\mathcal{H}$. Any realization of $\mathcal{H}$ other than $\mathcal{S}_1$ and $\mathcal{S}_2$ is an *irregular* separating ℓ-set. The two separating ℓ-sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are *regular*.

3.3 Wheel

A set of at least three separating ℓ-sets, $\ell \geq 2$, with a possible common intersection is a *wheel* [CBKT93]. A wheel can be represented by the set of vertices $\mathcal{C} \cup \{W_0, W_1, \ldots, W_{q-1}\}$ which satisfies the following conditions: (1) $q > 2$, (2) $|W_i| = |W_j|$, (3) $\forall i \neq j$, $\mathcal{C} \cup W_i \cup W_j$ is a separating ℓ-set unless in the case that $j = ((i+1) \mod q)$, (4) each vertex in $\mathcal{C}$ is adjacent to a vertex in each of the connected components created by removing any separating ℓ-set in the wheel, and (5) $\forall j \neq (i+1) \mod q$, $\mathcal{C} \cup W_i \cup W_j$ is a degree-2 separating ℓ-set. The set of vertices $\mathcal{C}$ is the *center* of the wheel. Each set of vertices W_i is a *side* of the wheel. For more details, see [CBKT93, Kan93a]. Each separating set of the form $\mathcal{C} \cup W_i \cup W_j$, $i \neq j$, is a separating set *represented* by the given wheel.

The separating sets in a wheel can also be characterized as follows. Two separating sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are *crossing each other* if $\mathcal{S}_1$ separates two vertices in $\mathcal{S}_2$ and vice versa. We define a separating set $\mathcal{C} \cup W_i \cup W_j$ in a wheel is a *crossing separating set* if $i \neq j$ and $j \neq (i+1) \mod q$. Every two distinct crossing separating sets in a wheel are crossing each other. It is worthwhile noting that there is no edge between a vertex in W_i and a vertex in

W_j if $j \neq (i+1) \mod q$. The separating degree of any crossing separating set in a wheel is two. We define the *unit size* of the wheel to be ℓ. For $\ell = 2$, a wheel is also called a *polygon* [Tut66]. Using a wheel, a total of $O(h^2)$ separating sets can be represented in $O(h)$ space.

3.4 Block

Given a proper subset of vertices $\mathcal{B}$ in $G = (V, E)$, the *border* of $\mathcal{B}$, $\textsc{border}(\mathcal{B}, G)$, is $\{u \mid u \in \mathcal{B}, u \text{ is adjacent to a vertex in } V \setminus \mathcal{B}\}$. The border of V in G is $\emptyset$. The *neighbor* of $\mathcal{B}$ in G, $\textsc{neighbor}(\mathcal{B}, G)$, is $\{u \mid u \in V \setminus \mathcal{B}, u \text{ is adjacent to a vertex in } \mathcal{B}\}$. The neighbor of V in G is $\emptyset$. If G is k-vertex-connected and $\mathcal{U}$ is subset of vertices of V, then $|\textsc{border}(\mathcal{U}, G)| \geq \min\{k, |\mathcal{U}|\}$ and $|\textsc{neighbor}(\mathcal{U}, G)| \geq \min\{k, |V| - |\mathcal{U}|\}$.

An *ℓ-block* in G is a maximal set of vertices $\mathcal{B}$ such that there are at least ℓ internally vertex-disjoint paths between every two distinct vertices in $\mathcal{B}$. By definition, a vertex with degree ℓ in G is an x-block, for all $x > \ell$. The set of vertices in a graph is a 0-block. The set of vertices in a connected component is a 1-block.

A subset of vertices $\mathcal{B}$ in $G = (V, E)$ is a *special block* for reaching k-vertex-connectivity if (1) $\mathcal{B} \subseteq \mathcal{S}$, where $\mathcal{S}$ is a separating set with $|\mathcal{S}| < k$, (2) $|\textsc{neighbor}(\mathcal{B}, G)| < k$, (3) there is no $Q \subset \mathcal{B}$ with $|\textsc{neighbor}(Q, G)| < k$, and (4) $\textsc{neighbor}(\mathcal{B}, G) \cup \mathcal{B} \neq V$.

The following claim and corollary state properties related to the border of a special block.

Claim 3.1 *For any subset of vertices H in a separating set,* $\textsc{border}(H, G) = H$.

Proof: Let $H \subseteq \mathcal{S}$, where $\mathcal{S}$ is a separating set. if $\textsc{border}(H, G) \neq H$, then these is a vertex $u \in H$ such that $u \notin \textsc{border}(H, G)$. Thus $\mathcal{S} \setminus \{u\}$ separates every two vertices that can be separated by $\mathcal{S}$. Hence there are at most $|\mathcal{S}| - 1$ internally vertex-disjoint paths between every two vertices separated by $\mathcal{S}$. Thus $\mathcal{S}$ is not a separating set. □

Corollary 3.2 *If $\mathcal{B}$ is a special block in G for reaching k-vertex-connectivity,* $\textsc{border}(\mathcal{B}, G) = \mathcal{B}$.

Proof: Since a special block is a subset of a separating set, this corollary follows from Claim 3.1. □

The structure and whereabouts of k-blocks in G are crucial in augmenting G to reach k-vertex-connectivity using the smallest number of edges, since adding any edge between a pair of vertices in a k-block is redundant. Note that the set of all vertices in a k-vertex-connected graph is a k-block.

3.5 Block Graph

Given a graph $G = (V, E)$, we construct its 4-block graph, 4-$\text{BLK}(G)$ as follows. We first find the 2-block graph, which is a forest, for G [RG77]. The set of vertices in the 2-block graph are the set of cut edges, cutpoints and 2-blocks in G. There is an edge between a 2-block $\mathcal{B}_2$ and a cutpoint c, if $c \in \mathcal{B}_2$. There is an edge between a cut edge e and a cutpoint c if c is an endpoint of e. If $|\mathcal{B}_2| > 1$, then the induced subgraph of G on $\mathcal{B}_2$, $G_{\mathcal{B}_2}$, is biconnected. For each 2-block $\mathcal{B}_2$, we find its 3-block tree on $G_{\mathcal{B}_2}$ [HT73] (see also [HR91]). In the 3-block graph, a separating pair is either represented as a vertex or is represented as two vertices in a polygon. Given a separating hybrid-pair $\mathcal{H}$ in the 3-block graph, if $\mathcal{H}$ consists of two edges, then $\mathcal{H}$ is represented in a polygon. Each 3-block is also represented as a vertex. The details of a 3-block tree can be found in [HR91].

Given a nontrivial 3-block, the set of vertices in a *Tutte component* [Tut66] is triconnected. A Tutte component for a 3-block $\mathcal{B}_3$ can be obtained by adding a *virtual edge* between the two vertices in each separating pair $\mathcal{S}_2$, $\mathcal{S}_2 \in \mathcal{B}_3$, to $G_{\mathcal{B}_3}$ — the induced subgraph of G on $\mathcal{B}_3$. For each non-trivial 3-block, we find the 4-block tree for its Tutte component [Hsu92]. In the 4-block tree for $\mathcal{B}_3$, each separating triplet in $\mathcal{B}_3$ is represented (1) as a vertex, (2) as three vertices in a wheel, or (3) as the neighbor of a special block. Given a separating hybrid-triplet $\mathcal{H}$, if $\mathcal{H}$ consists of more than one edge, then $\mathcal{H}$ is either represented in a wheel or is represented by its two regular realizations.

The collection and relation of the above 2-block graph, the 3-block forest, and the 4-block forest is 4-$\text{BLK}(G)$.

3.6 Augmentation Number

Given G, the smallest number of edges needed to add to G such that the resulting graph is k-vertex-connected is the *smallest k-vertex-connectivity augmentation number* $\text{AUG}_k(G)$. The set of edges added is a *smallest k-vertex-connectivity augmentation.* It is well-known that $\text{AUG}_1(G) = \text{COM}(G) - 1$. Simple formulas for computing $\text{AUG}_2(G)$ and $\text{AUG}_3(G)$ are given in [HR91, RG77]. For the case of G is triconnected, a simple formula for computing $\text{AUG}_4(G)$ is given in [Hsu92]. In this paper, we will give a formula to compute $\text{AUG}_4(G)$ for an arbitrary undirected graph. We will also give an efficient algorithm for finding a smallest four-connectivity augmentation.

4 Properties of Blocks

In this section, we examine properties of blocks. An ℓ-block $\mathcal{B}$ and a separating set $\mathcal{S}$ are *adjacent* if (1) $|\mathcal{S}| < \ell$, (2) every vertex $u \in \mathcal{S}$, it is either the case that $u \in \mathcal{B}$ or the case that u is adjacent to a vertex in $\mathcal{B}$, and (3) there is no separating set $\mathcal{S}'$ with $|\mathcal{S}'| < \ell$, $\mathcal{S} \neq \mathcal{S}'$, and $\mathcal{S}'$ separates a vertex in $\mathcal{S} \setminus \mathcal{S}'$ and a vertex in $\mathcal{B}$.

Given a subset of vertices $\mathcal{B}$ in a graph, $|\mathcal{B}| > 1$, k is an *order* of $\mathcal{B}$ if there are k internally vertex-disjoint paths between every two vertices in $\mathcal{B}$. The *largest order* of $\mathcal{B}$ is the largest integer k such that k is an order of $\mathcal{B}$. Two subsets of vertices $\mathcal{B}_1$ and $\mathcal{B}_2$ are of *the same order* if there is an integer w such that w is an order of both $\mathcal{B}_1$ and $\mathcal{B}_2$.

4.1 Degree and Demand of a Block

We first give definitions that are useful in computing the minimum number of edges needed to add to each ℓ-block, for all $\ell < k$, such that the resulting graph contains exactly one ℓ-block, for each $\ell < k$, i.e., the resulting graph is k-vertex-connected.

Definition 4.1 *Let $\mathcal{B}$ be a subset of vertices in $G = (V, E)$ and let $\ell_{\mathcal{B}}$ be the largest order of $\mathcal{B}$ if $|\mathcal{B}| > 1$. Then* $\textsc{vdeg}(\mathcal{B}, G) = \textsc{neighbor}(\mathcal{B}, G)$ *if* $|\mathcal{B}| = 1$ *and* $\textsc{vdeg}(\mathcal{B}, G) = \{u \mid u \text{ is in a separating set that is adjacent to } \mathcal{B}\}$ *if $|\mathcal{B}| > 1$. The* **degree** *of $\mathcal{B}$ in G is* $\textsc{degree}(\mathcal{B}, G) = |\textsc{vdeg}(\mathcal{B}, G)|$.

Definition 4.2 *Let $\mathcal{B}$ be a subset of vertices in $G = (V, E)$ where $|V| > k$ and let $\ell_{\mathcal{B}}$ be the largest order of $\mathcal{B}$ if $|\mathcal{B}| > 1$. The* **demand** *of $\mathcal{B}$ such that G can reach k-vertex-connectivity is*

$$\textsc{demand}_k(\mathcal{B}, G) = \begin{cases} \max\{k - \textsc{degree}(\mathcal{B}, G), 0\} & \textit{if } |\mathcal{B}| = 1 \textit{ or } \mathcal{B} \\ & \textit{is a special block,} \\ \sum_{\forall R} \textsc{demand}_k(R, G) & \textit{if } 1 < |\mathcal{B}| < k \textit{ and} \\ & \mathcal{B} \textit{ is a separating set,} \\ 0 & \textit{if } \mathcal{B} = V \textit{ and} \\ & \mathcal{B} \textit{ is a } k\textit{-block,} \\ \max\{k - \textsc{degree}(\mathcal{B}, G) + \sum_{\forall Q} \textsc{demand}_k(Q, G), & \textit{if } \mathcal{B} \neq V \textit{ and} \\ 0\} & \mathcal{B} \textit{ is a } k\textit{-block,} \\ \max\{k - \textsc{degree}(\mathcal{B}, G) + \sum_{\forall Q} \textsc{demand}_k(Q, G), & \\ \sum_{\forall H} \textsc{demand}_k(H, G) + \sum_{\forall T} \textsc{demand}_k(T, G)\} & \textit{otherwise,} \end{cases}$$

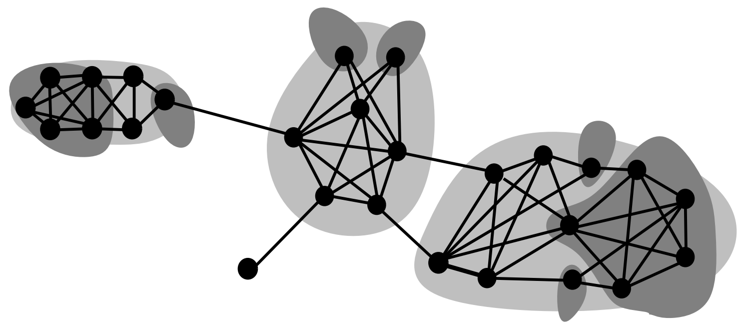

Figure 2: Illustrating the degree and the demand of a subset of vertices in a graph. In G, $\text{DEGREE}(A, G) = 1$ and $\text{DEMAND}_4(A, G) = 4$; $\text{DEGREE}(B, G) = 4$ and $\text{DEMAND}_4(B, G) = 2$; $\text{DEGREE}(C, G) = 2$ and $\text{DEMAND}_4(C, G) = 3$; $\text{DEGREE}(D_i, G) = 3$ and $\text{DEMAND}_4(D_i, G) = 1$, $1 \leq i \leq 5$; $\text{DEGREE}(H, G) = 3$ and $\text{DEMAND}_4(H, G) = 1$; $\text{DEGREE}(I, G) = 2$ and $\text{DEMAND}_4(I, G) = 2$.

where R is a special block in $\mathcal{B}$, Q is a special block in $\mathcal{B}$ that is contained in a separating r-set, $r < \ell_{\mathcal{B}}$, H is an $(\ell_{\mathcal{B}} + 1)$-block in $\mathcal{B}$ that is not contained in separating $\ell_{\mathcal{B}}$-sets, and T is a special block in $\mathcal{B}$ that is not contained in any $(\ell_{\mathcal{B}} + 1)$-block in $\mathcal{B}$.

If G is k-vertex-connected, then $\text{DEMAND}_k(\mathcal{B}, G) = 0$. The *demand* of G for reaching k-vertex-connectivity is $\text{DEMAND}_k(V, G)$. Recall that if $\mathcal{B}$ is a special block for reaching k-vertex-connectivity, then $\text{BORDER}(\mathcal{B}, G) = \mathcal{B}$ (Corollary 3.2). Intuitively, given a subset of vertices $\mathcal{B}$ with $\text{VDEG}(\mathcal{B}, G) \cup \mathcal{B} \subset V$, $\text{VDEG}(\mathcal{B}, G)$ separates a vertex in $\mathcal{B} \setminus \text{VDEG}(\mathcal{B}, G)$ from a vertex in $V \setminus (\mathcal{B} \cup \text{VDEG}(\mathcal{B}, G))$. The demand of $\mathcal{B}$ is the minimum number of edges needed to add to $\mathcal{B}$ such that the degree of $\mathcal{B}$ is at least k in the resulting graph, i.e., the demand of $\mathcal{B}$ becomes zero.

By the above definition, $\text{DEMAND}_k(V, G) \geq \text{DEMAND}_k(\mathcal{S}, G) + \text{DEMAND}_{k-|\mathcal{S}|}(V \setminus \mathcal{S}, G - \mathcal{S})$ for any separating set $\mathcal{S}$ with cardinality less than k. An example for the degree and the demand of a subset of vertices in a graph is illustrated in Figure 2.

Let $G = (V, E)$ be a $(k-1)$-vertex-connected graph. To increase the vertex-connectivity of G by one, $\text{DEMAND}_k(V, G)$ equals to the number of k-blocks that are leaves in its k-block tree, for every $k \leq 4$ [ET76, HR91, HR93, Hsu92]. For $k \leq 3$ and G is not $(k-1)$-vertex-connected, our definition is equivalent to similar definitions given in [ET76, HR91, WN93].

4.2 Demanding Vertices in a Block

We now give a definition to identify vertices in an ℓ-block $\mathcal{B}_\ell$ to whom adding new edges increases the degree of $\mathcal{B}_\ell$ (i.e., decreases the demand of $\mathcal{B}_\ell$).

Definition 4.3 *Let $\mathcal{B}$ be a subset of vertices in $G = (V, E)$ with* $\textsc{demand}_k(\mathcal{B}, G) > 0$. *Let u be a vertex in $\mathcal{B}$. If there is a vertex v such that* $\textsc{demand}_k(\mathcal{B}, G) > \textsc{demand}_k(\mathcal{B}, G \cup \{(u, v)\})$, *then u is a* **demanding vertex** *in $\mathcal{B}$ for reaching k-vertex-connectivity.*

Note that if $\mathcal{B}$ is a special block in G for reaching k-vertex-connectivity, then by Corollary 3.2 $\textsc{border}(\mathcal{B}, G) = \mathcal{B}$. By the definition of a special block, $\textsc{neighbor}(\mathcal{B}, G) \cup \mathcal{B} \subset V$. Thus every vertex in $\mathcal{B}$ is a demanding vertex in $\mathcal{B}$ for making G k-vertex-connected.

We describe in the following claims, the intersection and inclusive relations of various blocks. We first describe the intersection of two ℓ-blocks, $\ell \leq 4$.

Claim 4.4 *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two ℓ-blocks, $\ell \leq 4$. Then $\mathcal{B}_1 \cap \mathcal{B}_2$ is a subset of a separating set that is adjacent to both $\mathcal{B}_1$ and $\mathcal{B}_2$ and whose cardinality is less than ℓ. Furthermore, $\mathcal{B}_1 \cap \mathcal{B}_2$ is in every separating set which separates a vertex in $\mathcal{B}_1 \setminus \mathcal{B}_2$ and a vertex in $\mathcal{B}_2 \setminus \mathcal{B}_1$.*

Proof: This claim follows from the structural descriptions of all 2-blocks [Har69], 3-blocks [HT73, Tut66], and 4-blocks [Hsu92, KR91, KTDBC91]. □

Claim 4.5 *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two ℓ-blocks, $\ell \leq 4$. Let $\mathcal{I} = \mathcal{B}_1 \cap \mathcal{B}_2$. For reaching k-vertex-connectivity, $k \leq 4$, there is no special block in $\mathcal{I}$.*

Proof: If $\ell \leq 1$, then $\mathcal{I} = \emptyset$. Thus we assume that $\ell > 1$. We also know that if $\ell = 4$, then there are at least four internally vertex-disjoint paths between every two vertices in $\mathcal{B}_i$, $i \in \{1, 2\}$. By Menger's theorem, the degree of any subset of vertices in $\mathcal{I} \subset \mathcal{B}_i$ is at least four. Thus there is no special block in $\mathcal{I}$. Hence we only have to prove the claim when ℓ is two or three. By Claim 4.4, $\mathcal{I}$ is part of a separating set with cardinality less than ℓ.
Case 1: $\ell = 2$. Thus $\mathcal{I} = \{u\}$, where u is a vertex. Because $\mathcal{B}_1$ and $\mathcal{B}_2$ are 2-blocks, u must be adjacent to at least two vertices in $\mathcal{B}_1 \setminus \mathcal{B}_2$ and to at least two vertices in $\mathcal{B}_2 \setminus \mathcal{B}_1$. The degree of u is at least four. Thus $\{u\}$ cannot be a special block for reaching k-vertex-connectivity.
Case 2: $\ell = 3$. Let $H \subseteq \mathcal{I}$ be a subset of vertices with degree less than four. Then either $|\textsc{neighbor}(H, G) \cap \mathcal{B}_1| = 1$ or $|\textsc{neighbor}(H, G) \cap \mathcal{B}_2| = 1$. Without loss of generality, assume that $|\textsc{neighbor}(H, G) \cap \mathcal{B}_1| = 1$. Let $X = (\textsc{neighbor}(H, G) \cap \mathcal{B}_1) \cup (\mathcal{I} \setminus H)$. Thus $|X| \leq 2$ and X is a separating set which separates a vertex in H and a vertex in $\mathcal{B}_1 \setminus H$. This contradicts the fact that $\mathcal{B}_1$ is a 3-block. □

The following claim and corollary lead to a lemma for lower bounding the smallest augmentation number.

Claim 4.6 *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two distinct ℓ-blocks in G, $\ell \leq 4$. For making G four-connected, if $\mathcal{B}_2 \not\subseteq \mathcal{B}_1$ and $\mathcal{B}_1 \not\subseteq \mathcal{B}_2$, then there is no intersection between the set of demanding vertices in $\mathcal{B}_1$ and the set of demanding vertices in $\mathcal{B}_2$.*

Proof: Let $\mathcal{I} = \mathcal{B}_1 \cap \mathcal{B}_2$. If $\mathcal{I} = \emptyset$, then the claim is obviously true. Thus we assume that $\mathcal{I}$ is non-empty. By Claim 4.4, $\mathcal{I} \subseteq \text{VDEG}(\mathcal{B}_i, G)$. Thus there is no demanding vertex of B_i, $i \in \{1, 2\}$, in $\mathcal{I}$. □

Corollary 4.7 *Let u_1 and u_2 be two vertices in $G = (V, E)$. Then $\text{DEMAND}_4(V, G \cup \{(u_1, u_2)\}) \geq \text{DEMAND}_4(V, G) - 2$.*

Proof: By the definition of the demand of G in order to decrease the demand, the degree of vertices should be increased. Note that by adding the edge (u_1, u_2), the degrees of u_1 and u_2 both increase by one. Let w be an integer that is either one or two.
Case 1: u_w is not a demanding vertex for any block (special or non-special) in the graph. Increasing the degree of u_w does not decreasethe demand.
Case 2: u_w is a demanding vertex for a special block. By definition of a special block and Claim 4.5, u_w cannot be in any other special block. Also by the way the demand of the graph is computed, the demand of the graph is decreased by at most one by increasing the degree of u_w.
Case 3: u_w is a demanding vertex for an ℓ-block. By Claim 4.6, u_w cannot be a demanding vertex for any other ℓ-block. Thus the demand of the graph is decreased by at most one by increasing the degree of u_w.

Note that by adding an edge, several blocks of the same order x, $1 \leq x \leq 4$, that are in the same $(x-1)$-block $\mathcal{B}$ can be merged into an x-block. However, the demand of $\mathcal{B}$ can be decreased by at most two according to the way the demand is computed and the above discussion. The demands of those blocks not containing an endpoint of the added edge or that are not merged stay the same. For reaching four-connectivity, the demand of the graph is decreased by at most two by adding the edge (u_1, u_2). □

Lemma 4.8 *We need to add at least $\left\lceil \frac{\text{DEMAND}_4(V,G)}{2} \right\rceil$ edges to $G = (V, E)$ in order to raise the vertex-connectivity of G to four.*

Proof: Follows from Corollary 4.7. □

5 Properties of Separating Sets

In this section, we examine properties of separating sets related to the smallest augmentation of a graph for reaching k-vertex-connectivity.

5.1 Fundamental Properties

Classification of Components

A connected component H is *created* by removing a separating set $\mathcal{S}$ from G if H is a connected component in $G-\mathcal{S}$, but H is properly contained in a connected component H' of G and H' contains a vertex that is not in $\mathcal{S} \cup V_H$. We first describe an important definition to classify components created by removing separating sets.

Definition 5.1 *Given a set of separating sets $\{\mathcal{S}_1, \ldots, \mathcal{S}_w\}$, an* ***x*-component** *H, is a connected component in $G - \cup_{i=1}^{w} \mathcal{S}_i$, with the property that H is also a connected component in $G - \mathcal{S}'$, where (1) $\mathcal{S}' \subseteq \mathcal{S}_j$, for some $1 \leq j \leq w$, (2) H is created by removing $\mathcal{S}'$, (3) $x = |\mathcal{S}'|$, and (4) x is the smallest integer satisfying the above conditions.*

For example, in $G - \{1,2,3\}$ in Figure 1, $G_1 = (\{5\}, \emptyset)$ is a 1-component, $G_2 = (\{6\}, \emptyset)$ is a 2-component, and $G_3 = (\{7\}, \emptyset)$ is a 3-component. A connected component not containing any vertex in the set of removed separating sets is a 0-component. Recall from the definition of separating sets, for every separating ℓ-set $\mathcal{S}$ in G, there are at least two ℓ-components in $G - \mathcal{S}$.

Given a set of separating sets $S = \{\mathcal{S}_1, \ldots, \mathcal{S}_w\}$, let $V_S = \cup_{i=1}^{w} \mathcal{S}_i$. The *connection* of a connected component H in $G - V_S$ is the number of vertices in V_S that are adjacent to (in G) a vertex in H. Thus the connection of an x-component is x.

Demand and Component

Let H be a y-component created by removing a separating x-set, $x \geq y$, $\mathcal{S}$ from $G = (V, E)$. Let $U \subseteq \mathcal{S}$ where $u \in U$ implies that there is a $(y+1)$-block $\mathcal{B} \subseteq V_H \cup \mathcal{S}$ and $u \in \mathcal{B}$. We define $\Phi(H) = U \cup V_H$. The *contribution* of H for computing $\text{DEMAND}_k(V, G)$ is $\text{DEMAND}_k(\Phi(H), G)$. It is also denoted as H *contributes* $\text{DEMAND}_k(\Phi(H), G)$ in computing $\text{DEMAND}_k(V, G)$. Note that $\text{DEMAND}_k(\Phi(H), G) \geq \text{DEMAND}_{k-|\mathcal{S}|}(V_H, G - \mathcal{S})$. The following claims and corollary state the relation between the demand of the graph and each of the components created by removing a separating set.

Claim 5.2 *Let $\mathcal{S}$ be a separating ℓ-set in G, $\ell \leq 3$. Let $G_1 = (V_1, E_1)$ be an ℓ-component in $G - \mathcal{S}$. Then there is an $(\ell+1)$-block $B_{\ell+1}$ where (i) $\mathcal{B}_{\ell+1} \subseteq (V_1 \bigcup \mathcal{S})$, and (ii) $\text{DEGREE}(\mathcal{B}_{\ell+1}, G) \leq \ell$.*

Proof: We prove this claim is true for any value of ℓ up to three using induction on the value of $|V_1|$. If $|V_1| = 1$, then V_1 is an $(\ell+1)$-block with degree ℓ. Thus this claim is true for any value of ℓ up to three. For induction hypothesis, we assume that this claim is true for any G_1 with at most s vertices, where s is an integer and $s \geq 1$. We now prove that this claim is true G_1 with $s+1$ vertices. If there is an $(\ell+1)$-block R with $V_1 \subseteq R$, then $\text{VDEG}(R, G) = \mathcal{S}$. Thus $\text{DEGREE}(R, G) = \ell$. Hence the claim is true. Assume that there are at most ℓ internally vertex-disjoint paths between two vertices in V_H. Thus we can find a separating set $\mathcal{S}'$, $|\mathcal{S}'| \leq \ell$, to separate two vertices in V_H. If $\mathcal{S}$ and $\mathcal{S}'$ are not crossing each other, then there is an $|\mathcal{S}'|$-component $G_1' = (V_1', E_1')$ in $G - \mathcal{S}'$, and $V_1' \subset V_H$. Thus we can apply the induction hypothesis. Assume that $\mathcal{S}$ and $\mathcal{S}'$ are crossing each other. Let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ where $\mathcal{S}'$ separates $\mathcal{S}_1$ and $\mathcal{S}_2$, and let $\mathcal{S}_1 \subseteq V_1$. Let $\mathcal{S}' = \mathcal{S}_1' \cup \mathcal{S}_2'$ where $\mathcal{S}$ separates $\mathcal{S}_1'$ and $\mathcal{S}_2'$, and let $\mathcal{S}_1' \subseteq V_1$. Then $\mathcal{S}_1 \cup \mathcal{S}_1'$ and $\mathcal{S}_2 \cup \mathcal{S}_1'$ are separating sets. Since $\ell \leq 3$, either $|\mathcal{S}_1 \cup \mathcal{S}_1'| \leq 3$ or $|\mathcal{S}_2 \cup \mathcal{S}_1'| \leq 3$. Assume without loss of generality, let $\mathcal{S}'' = \mathcal{S}_1 \cup \mathcal{S}_1'$ and $|\mathcal{S}''| \leq 3$. Thus there is an $|\mathcal{S}''|$-component $G_1'' = (V_1'', E_1'')$ in $G - \mathcal{S}''$ and $V_1'' \subset V_H$. Thus we may apply the induction hypothesis to prove this claim. Hence the claim holds. □

Claim 5.3 *Given a separating set $\mathcal{S}$ in $G = (V, E)$, each ℓ-component, $\ell \leq |\mathcal{S}|$, in $G - \mathcal{S}$ contributes at least $k - \ell$ in computing $\text{DEMAND}_k(V, G)$, $k \leq 4$.*

Proof: Let H be an ℓ-component in $G - \mathcal{S}$. Let V_H be the set of vertices in H. Assume that H is created by removing $\mathcal{S}'$ where $\mathcal{S}' \subseteq \mathcal{S}$ and $|\mathcal{S}'| = \ell$. By properties of the 4-block graph and Claim 5.2, there is an $(\ell+1)$-block with degree ℓ in $V_H \cup \mathcal{S}'$. Thus it contributes at least $k - \ell$ in computing $\text{DEMAND}_k(V, G)$. □

Corollary 5.4 *Let S be a set of separating sets in $G = (V, E)$. For making G k-vertex-connected, $k \leq 4$, every connected component in $G - V_H$ with connection ℓ contributes at least $k - \ell$ in computing $\text{DEMAND}_k(V, G)$ if $\ell < k$.*

Proof: This is a corollary of Claim 5.3. □

We then state a claim to quantify the number of various components after removing a set of up to three separating sets with the same cardinality.

Claim 5.5 *Let $\mathcal{S}_i$, $1 \leq i \leq 3$, be separating ℓ-sets, $\ell \leq 3$, in G. Let $w_{i,j}$ be the number of j-components in $G - \mathcal{S}_i$, $1 \leq i \leq 3$. Given any $1 \leq r \leq 3$ and $1 \leq h \leq r$, in $G - \cup_{i=1}^{r} \mathcal{S}_i$, there are $(\sum_{i=1}^{r} w_{i,3}) - x_3$ 3-components, $(w_{h,2} - x_2)$ 2-components not containing the above 3-components, and $(w_{h,1} - x_1)$ 1-components not containing the above 2- or 3-components where $\sum_{i=1}^{3} x_i = r - 1$.*

Proof: We first prove the claim for the case $r = 2$. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are in the same 3-block, then there are at least $(w_{1,3} + w_{2,3} - 1)$ 3-components.

If they are in the same 2-block, but not in the same 3-block, then there are $(w_{1,3} + w_{2,3})$ 3-components. In this case, if, furthermore, $\mathcal{S}_2$ is in a 2-component in $G - \mathcal{S}_1$, then there are $(w_{1,2} - 1)$ 2-components.

If they are in the same 1-block, but not the same 2-block, then there are $(w_{1,3} + w_{2,3})$ 3-components. If, furthermore, $\mathcal{S}_2$ is in a 2-component (respectively, 1-component) in $G - \mathcal{S}_1$. Then there are either $(w_{1,2} - 1)$ 2-components (respectively, $(w_{1,1} - 1)$ 1-components). The case for $r = 3$ can be proved using a similar strategy. □

Separating Degree and Dividing Degree

Given a subset of vertices $\mathcal{S}$, the *separating degree* $\text{SD}(\mathcal{S}, G)$ is the number of $|\mathcal{S}|$-components in $G - \mathcal{S}$ if $\mathcal{S}$ is a separating set. For convenience, if $\mathcal{S}$ is not a separating set, $\text{SD}(\mathcal{S}, G) = 1$. The *dividing degree* $\text{DD}(\mathcal{S}, G)$ is $\text{COM}(G - \mathcal{S}) - \text{COM}(G) + 1$, i.e., the number of connected components created by removing $\mathcal{S}$. The separating degree of any separating set is at least two. The dividing degree of a separating set is greater than or equal to its separating degree. Intuitively, the separating degree of a separating set is the number of connected components that cannot be separated by removing a proper subset of $\mathcal{S}$. For example, in Figure 1, the separating degree of the separating triplet {2, 3, 4} is two and its dividing degree is five, while there are seven connected components in the resulting graph obtained by removing {2, 3, 4}.

The following claim and corollary are extended from a claim given in [Hsu92] and describe the properties of the degree of any subset of vertices in a separating set.

Claim 5.6 *[Hsu92] Every vertex u in a separating set $\mathcal{S}$ is adjacent to (in G) a vertex in each of the $|\mathcal{S}|$-components in $G - \mathcal{S}$.*

Proof: Assume that u is not adjacent to an $|\mathcal{S}|$-component H in $G - \mathcal{S}$. Thus H is also a connected component in $G - (\mathcal{S} \setminus \{u\})$. Then H is not an $|\mathcal{S}|$-component. □

Corollary 5.7 *The degree of any subset of vertices in a separating set $\mathcal{S}$ is at least equal to the separating degree of $\mathcal{S}$.*

Proof: Follows from Claim 5.6. □

Corollary 5.8 *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two distinct separating ℓ-sets. The degree of any subset of vertices in $\mathcal{S}_1 \cap \mathcal{S}_2$ is at least $\text{SD}(\mathcal{S}_1, G) + \text{SD}(\mathcal{S}_2, G) - 1$.*

Proof: This is a corollary of Claim 5.6 and Claim 5.5. □

We now state a claim with regard to the creation of new separating sets when adding edges. We will show that by adding edges properly, no separating set with undesirable properties (e.g., with separating degree greater than two) is created.

Claim 5.9 *Let κ be the vertex-connectivity of $G = (V, E)$. Let u and v be two vertices in V, and let $G' = G \cup \{(u, v)\}$. (1) Every separating κ-set in G' is also a separating κ-set in G,* *i.e., adding an edge creates no new separating κ-set.* (2) *Adding an edge creates only new separating ℓ-sets, $\ell > \kappa$, whose separating degrees are two.* (3) *If u and v are both not in any separating ℓ-set, then the separating degree of any separating ℓ-set in G does not increase by adding (u, v).*

Proof: Note that all separating sets in G and G' must be of cardinality at least κ.

Part (1): Adding an edge in a graph does not decrease the number of internally vertex-disjoint paths between every two vertices. If $\mathcal{S}$ is a separating κ-set in G', let G_1 be a connected component in $G' - \mathcal{S}$. If both u and v are in G_1, then $\mathcal{S}$ is a separating κ-set in G. The above is also true if both u and v are in $\mathcal{S}$. If u is in G_1 and v is in $\mathcal{S}$ (or vice versa), then $\mathcal{S}$ is a separating set with cardinality at most κ in G. Thus the claim holds.

Part (2): Let $\mathcal{S}$ be a separating ℓ-set in G', $\ell > \kappa$, but not in G. If $\mathrm{SD}(\mathcal{S}, G') > 2$, then $\mathrm{SD}(\mathcal{S}, G) \geq 2$. Thus $\mathcal{S}$ is also a separating set in G.

Part (3): Without connecting a connected component to a separating set $\mathcal{S}$, the separating degree of $\mathcal{S}$ does not increase. □

Components that are not Complicated

A separating set with cardinality less than $k-1$ is *trivial* for reaching k-vertex-connectivity if its dividing degree is two and it contains no special block. A separating set with cardinality less than $k-1$ is *non-trivial* for reaching k-vertex-connectivity if it is not trivial.

Let H be a connected component in $G - \mathcal{S}$, where $\mathcal{S}$ is a separating set. Let R be the set of separating sets in $\Phi(H)$ that are not proper super sets of $\mathcal{S}$. Let B be the set of blocks in $\Phi(H)$ that are not adjacent to $\mathcal{S}$. The component H is *simple* for reaching k-vertex-connectivity if (1) the dividing degree of every separating i-set, $i < k-1$, in R, is two, (2) there is at most one i-block in B with degree less than i for every $1 < i < k-1$, and (3) if there is an i-block B_i with degree less than i and a j-block B_j with degree less than j, then B_i properly contains B_j, (4) if there is a separating $(k-1)$-set in R with dividing degree more than two, then it is in a $(k-2)$-block with degree less than $k-2$. If H is not simple, then H is *complicated*. Let R' be the set of separating sets in R with cardinality less than $k-1$ and let R'' be the set of separating $(k-1)$-sets in R with dividing degree more than two. A simple component has the property that there are two vertices u and v in $\Phi(H)$ such that every separating set in $R' \cup R''$ separates u and v.

In the following three claims, let $\mathcal{S}$ be a separating ℓ-set in $G = (V, E)$ and let H be an h-component in $G - \mathcal{S}$. Let R be the set of separating sets in $\Phi(H)$ with cardinality less than k which are not proper super sets of $\mathcal{S}$. Let $\mathcal{S}' \in R$. Let B be the set of blocks in $\Phi(H)$ that are not adjacent to $\mathcal{S}$. The proofs of these lemmas can be easily deduced.

Claim 5.10 *For making G k-vertex-connected, $k \leq 4$, if $\mathrm{DEMAND}_k(\Phi(H), G) = \mathrm{DEMAND}_{k-\ell}(V_H, G - \mathcal{S})$, then the followings are true.* (1) *$h = \ell$.* (2) *H is simple.* (3) *$\mathcal{S}'$ is trivial if $r < k-1$.* (4) *There is no special block in $\mathcal{S}'$ if $r = k-1$ and $\mathrm{DD}(\mathcal{S}', G) > 2$.*

Claim 5.11 *For making G k-vertex-connected, $k \leq 4$, if* $\textsc{demand}_k(\Phi(H), G) \leq \textsc{demand}_{k-\ell}(V_H, G - \mathcal{S}) + 1$*, then the following are true. (1) H is simple. (2) There is at most one separating set in R with a special block. (3)* $\textsc{dd}(\mathcal{S}', G) = 2$ *if* $|\mathcal{S}'| < k - 1$.

Claim 5.12 *Let $B' = \{Y \mid Y \in B$ where the degree of Y is less than its smallest order$\}$. For making G k-vertex-connected, $k \leq 4$, if* $\textsc{demand}_k(\Phi(H), G) \leq \textsc{demand}_{k-\ell}(V_H, G - \mathcal{S}) + 2$*, then the following are true. (1) There is at most one separating set in R with dividing degree more than two and cardinality less than $k - 1$. (2) If there is a separating set in R with dividing degree more than two and cardinality less than $k - 1$, then the dividing degree of all other separating sets in R is two and there is no special block. (3) $|B'| \leq 2$ and if $|B'| = 2$, then the dividing degree of all other separating sets in R is two and there is no special block. (4) If there is a separating set in R with dividing degree more than two, then there is no special block.*

A connected component H in $G - \mathcal{S}$ with $\textsc{demand}_k(\Phi(H), G) = \textsc{demand}_{k-\ell}(V_H, G - \mathcal{S}) + r$ is an *r-simple* component.

5.2 Augmenting within a Separating Set

Let $\mathcal{S}$ be a subset of vertices in $G = (V, E)$. Then $a_k(\mathcal{S}, G)$ is the minimum number of edges needed to add such that in the resulting graph $|\textsc{neighbor}(F, G)| \geq \min\{k, |V \setminus F|\}$ for every $F \subseteq \mathcal{S}$. If $\mathcal{S}$ is a separating set with cardinality less than k, then $a_k(\mathcal{S}, G)$ is the minimum number of edges to add such that there is no special block in $\mathcal{S}$ for making G k-vertex-connected. It is obvious that $a_2(\{c\}, G) = 0$ for any cutpoint c, and $a_3(\mathcal{S}, G) \leq 1$ for any separating set $\mathcal{S}$ with cardinality at most two.

Claim 5.13 *Let $\mathcal{S}$ be a separating ℓ-set in G. For making G k-vertex-connected, $\ell < k$, the followings are true. (1)* $2 \cdot a_k(\mathcal{S}, G) \geq \textsc{demand}_k(\mathcal{S}, G) \geq a_k(\mathcal{S}, G)$*, $k \leq 4$. (2) $a_4(\mathcal{S}, G) \leq 3$. (3) If $\ell = 1$, then $a_4(\mathcal{S}, G) \leq 2$, $a_4(\mathcal{S}, G) = 2$ implies* $\textsc{sd}(\mathcal{S}, G) = 2$*, and $a_4(\mathcal{S}, G) = 1$ implies* $\textsc{sd}(\mathcal{S}, G) \leq 3$*. (4) If $\ell = 2$, then $a_4(\mathcal{S}, G) \leq 3$, $a_4(\mathcal{S}, G) > 0$ implies* $\textsc{sd}(\mathcal{S}, G) \leq 3$*, and $a_4(\mathcal{S}, G) > 1$ implies* $\textsc{sd}(\mathcal{S}, G) = 2$.

Proof: Part (1) is trivial since adding an edge to a special block reduces its demand by at most 1. There are two endpoints of an edge, thus this part follows. Part (2) is true for $\ell = 3$ [Hsu92]. We prove parts (3) and (4). For the case of $\ell = 1$, let $\mathcal{S} = \{u\}$. The degree of u is at least two (by the fact that the separating degree of any separating set is at least two and Corollary 5.7). Since $a_k(\{u\}, G) = k - \textsc{degree}(\{u\}, G)$, the claim is true. For the case of $\ell = 2$, let $\mathcal{S} = \{u, v\}$. The degrees of u and v are both at least two. If u is adjacent to v, then the degrees of u and v are at least three. Thus $a_4(\mathcal{S}, G) \leq 2$ and $a_4(\mathcal{S}, G) = 2$

if $\textsc{sd}(\mathcal{S},G)=2$. If u and v are not adjacent, then $a_4(\mathcal{S},G)\leq 3$, and $a_4(\mathcal{S},G)=3$ implies $\textsc{sd}(\mathcal{S},G)=2$. Note that $a_4(\mathcal{S},G)=1$ could imply either $\textsc{sd}(\mathcal{S},G)=2$ or $\textsc{sd}(\mathcal{S},G)=3$. It is also true that $a_4(\mathcal{S},G)=2$ implies $\textsc{sd}(\mathcal{S},G)=2$. □

5.3 Separation Constraint

Let $\mathcal{S}$ be a subset of vertices in $G=(V,E)$. If $|\mathcal{S}|<k$, the *separation constraint* of $\mathcal{S}$ for making G k-vertex-connected is $\textsc{sc}_k(\mathcal{S},G)=a_k(\mathcal{S},G)+\textsc{aug}_{k-|\mathcal{S}|}(G-\mathcal{S})$. The separation constraint for $\mathcal{S}$ is the number of edges needed to add to G such that in the resulting graph (1) there is no separating set $\mathcal{S}'\subseteq\mathcal{S}$, (2) there is no separating set $\mathcal{S}'$ with cardinality less than k and $\mathcal{S}\subset\mathcal{S}'$, and (3) for any $F\subseteq\mathcal{S}$, $\textsc{degree}(F,G)\geq k$ unless $F\cup\textsc{neighbor}(F,G)=V$. The separation constraint of a graph for reaching k-vertex-connectivity is the largest separation constraint among all separating sets with cardinality less than k.

Intuitively, if a graph is k-vertex-connected, then $G-\mathcal{S}$ is $(k-|\mathcal{S}|)$-vertex-connected for any subset of vertices $\mathcal{S}$ with cardinality less than k. There is also no subset of vertices in $\mathcal{S}$ with degree less than k.

Lemma 5.14 (1) *We need to add at least* $\textsc{sc}_k(\mathcal{S},G)$ *edges to* G *in order to make* G k*-vertex-connected.* (2) $\textsc{sc}_k(\mathcal{S},G\cup\{e\})\geq\textsc{sc}_k(\mathcal{S},G)-1$, *where* e *is an edge not in* G.

Proof: Note that if G is k-vertex-connected, then $G-\mathcal{S}$ must be $(k-|\mathcal{S}|)$-vertex-connected. Thus we must add at least $\textsc{aug}_{k-|\mathcal{S}|}(G-\mathcal{S})$ edges. It is also true that for all $F\subset\mathcal{S}$, either $F\cup\textsc{neighbor}(F,G)=V$ or $|\textsc{neighbor}(F,G)|>k$. Thus we must add at least $a_k(\mathcal{S},G)$ edges. Observing that if $a_k(\mathcal{S},G)>a_k(\mathcal{S},G\cup\{e\})$, then one of the endpoints of e must be in $\mathcal{S}$. Thus adding e does not decrease $\textsc{aug}_{k-|\mathcal{S}|}(G-\mathcal{S})$. Observing also that if $\textsc{aug}_{k-|\mathcal{S}|}(G-\mathcal{S})$ is decreased by adding an edge e', then both of its endpoints must not be in $\mathcal{S}$. Thus adding e' does not decrease $a_k(\mathcal{S},G)$. This proves the lemma. □

For augmenting a triconnected graph to reach four-connectivity, $\textsc{sc}_4(\mathcal{S},G)=a_4(\mathcal{S},G)+\textsc{sd}(\mathcal{S},G)-1$. This lower bound of the smallest four-connectivity augmentation number imposed by a separating triplet when the graph is triconnected is also given in [Hsu92].

The following claims state the way to reduce the separation constraint and the demand of a disconnected graph.

Claim 5.15 *Every non-special* ℓ*-block,* $0<\ell\leq 4$*, with degree less than* ℓ *properly contains a non-special* $(\ell+1)$*-block with degree less than* $\ell+1$*.*

Proof: This claim follows from the structure of the 4-block graph. □

Claim 5.16 *For making a disconnected graph k-vertex-connected, $2 \leq k \leq 4$, we can add an edge such that the demand of the graph is reduced by two and at the same time the separating constraint of the graph is reduced by one.*

Proof: Let G_1 and G_2 be two distinct connected components in G. By Claim 5.15, we can find a k-block $\mathcal{B}_i$, for all $i \in \{1,2\}$, in G_i with degree less than k and $\mathcal{B}_i$ is properly contained in a w-block with degree less than w, for all $1 \leq w < k$. Let u_i, for all $i \in \{1,2\}$, be a demanding vertex in $\mathcal{B}_i$. Note that u_i is not part of any separating set with cardinality less than k. From the definition of the demand of a graph, $\text{DEMAND}_k(V, G \cup \{(u_1,u_2)\}) = \text{DEMAND}_k(V,G) - 2$.

Let $G' = G \cup \{(u_1,u_2)\}$. Since $\text{COM}(G') = \text{COM}(G) - 1$, the separation constraint of any separating $(k-1)$-set is reduced by one. Let $\mathcal{S}$ be a separating set with cardinality less than $k-1$. By the way the demand is defined, $\text{DEMAND}_{k-|\mathcal{S}|}(V \setminus \mathcal{S}, G' - \mathcal{S}) = \text{DEMAND}_{k-|\mathcal{S}|}(V \setminus \mathcal{S}, G - \mathcal{S}) - 2$. Using the above and the fact that $\text{COM}(G') = \text{COM}(G) - 1$, $\text{AUG}_{k-|\mathcal{S}|}(G' - \mathcal{S}) = \text{AUG}_{k-|\mathcal{S}|}(G - \mathcal{S}) - 1$. Thus the separation constraint of $\mathcal{S}$ is decreased by one. $\Box$

Let $G' = G - \mathcal{S}$ and let V' be the set of vertices in G'. It is well-known that (1) $\text{AUG}_1(G') = \text{COM}(G') - 1$, and (2)

$$\text{AUG}_{k-|\mathcal{S}|}(G') = \max\{\left\lceil \frac{\text{DEMAND}_{k-|\mathcal{S}|}(V',G')}{2} \right\rceil, d_{k-|\mathcal{S}|-1} - 1\}$$

for all $k - |\mathcal{S}| \in \{2,3\}$ [HR91, HR93, RG77], where $d_{k-|\mathcal{S}|-1}$ is the maximum number of connected components obtained in G' by removing a separating $(k - |\mathcal{S}| - 1)$-set.

Let $r(n,k)$ be the minimum number of edges needed to augment an empty graph of n vertices to reach k-vertex-connectivity. It is well-known [Har62] that

$$r(n,k) = \begin{cases} n-1 & \text{if } k = 1; \\ \left\lceil \frac{n \cdot k}{2} \right\rceil & \text{otherwise.} \end{cases}$$

We know that $\text{AUG}_\ell(G) \geq r(\text{COM}(G), \ell)$. Thus we have the following simple corollary for a lower bound on the value of the separation constraint for a separating set.

Corollary 5.17 *Let $\mathcal{S}$ be a separating set with cardinality less than k in G. For making G k-vertex-connected,* (1) $\text{SC}_k(\mathcal{S},G) \geq a_k(\mathcal{S},G) + \text{COM}(G - \mathcal{S}) - 1$ *if* $|\mathcal{S}| = k-1$, *and* (2) $\text{SC}_k(\mathcal{S},G) \geq a_k(\mathcal{S},G) + \left\lceil \frac{(k-|\mathcal{S}|)\cdot \text{COM}(G-\mathcal{S})}{2} \right\rceil$ *if* $|\mathcal{S}| < k-1$.

Proof: Let $k' = k - |\mathcal{S}|$ and let $G' = G - \mathcal{S}$. Thus $k' \geq 1$. Since $\text{AUG}_{k'}(G') \geq r(\text{COM}(G'), k')$, this corollary holds. $\Box$

5.4 Critical, Massive, and Balanced

Let $\mathcal{S}$ be a separating set with cardinality less than k in $G = (V, E)$. For making G k-vertex-connected, if $|\mathcal{S}| < k$ and $\textsc{sc}_k(\mathcal{S}, G) > \left\lceil \frac{\textsc{demand}_k(V,G)}{2} \right\rceil$, then $\mathcal{S}$ is *massive*. If $|\mathcal{S}| < k$ and $\textsc{sc}_k(\mathcal{S}, G) = \left\lceil \frac{\textsc{demand}_k(V,G)}{2} \right\rceil$, then $\mathcal{S}$ is *critical*. A graph with no critical and massive separating set is *balanced*. For a balanced graph, it seems that the "most important job" in augmenting a graph is to make sure that the demand of the graph becomes zero. For an unbalanced graph, "taking care of" critical and massive separating sets seems to be more important than "taking care of" the demand of the graph. We now describe properties of critical and massive separating sets (e.g., their cardinalities and inter-relations) in the following paragraphs.

Claim 5.18 *Let $\mathcal{S}$ be a separating set in $G = (V, E)$. Let $V' = V \setminus \mathcal{S}$ and let $G' = G - \mathcal{S}$. For making G k-vertex-connected, $k \leq 4$, $\textsc{sc}_k(\mathcal{S}, G) = a_k(\mathcal{S}, G) + \left\lceil \frac{\textsc{demand}_{k-|\mathcal{S}|}(V',G')}{2} \right\rceil$ if $|\mathcal{S}| < k - 1$ and $\mathcal{S}$ is critical or massive.*

Proof: Let d_h be the maximum number of connected components obtained in G' by removing a separating h-set. Let $|\mathcal{S}| = \ell$. Let $k' = k - \ell$. Since $\mathcal{S}$ is critical or massive,

$$a_k(\mathcal{S}, G) + \textsc{aug}_{k'}(G') \geq \left\lceil \frac{\textsc{demand}_k(V,G)}{2} \right\rceil. \tag{1}$$

Note that $\textsc{aug}_{k'}(G') = \max\{d_{k'-1} - 1, \left\lceil \frac{\textsc{demand}_{k'}(V',G')}{2} \right\rceil\}$, since $k' \leq 3$. We will prove that $\textsc{aug}_{k'}(G') = \left\lceil \frac{\textsc{demand}_{k'}(V',G')}{2} \right\rceil$. In computing $\textsc{aug}_{k'}(G')$, assuming that

$$d_{k'-1} - 1 > \left\lceil \frac{\textsc{demand}_{k'}(V',G')}{2} \right\rceil.$$

Then $\textsc{aug}_{k'}(G') = d_{k'-1} - 1$. Thus we can obtain the following from Equation (1).

$$a_k(\mathcal{S}, G) + d_{k'-1} - 1 \geq \left\lceil \frac{\textsc{demand}_{k'}(V,G)}{2} \right\rceil. \tag{2}$$

Let $\mathcal{S}'$ be a separating set in G' whose removal makes the resulting graph contain $d_{k'-1}$ connected components. By Corollary 5.4, each connected component in $G' - \mathcal{S}'$ contributes at least $\ell + 1$ in computing $\textsc{demand}_k(V, G)$, except the one that is connected to $\mathcal{S}$ in G of which contributes at least one. There are at least two connected components in G'.

Since $G' = G - \mathcal{S}$ and $\mathcal{S}$ is a separating set, by Claim 5.3 each connected component in G' contributes at least k' in computing $\text{DEMAND}_k(V, G)$. Thus

$$\text{DEMAND}_k(V, G) \geq a_k(\mathcal{S}, G) + (\ell + 1) \cdot (d_{k'-1} - 1) + k - \ell + 1. \tag{3}$$

By substituting Equation (3) into Equation (2), we can obtain $a_k(\mathcal{S}, G) \geq 4$. This is a contradiction to parts (2), (3), and (4) in Claim 5.13. Thus the claim holds. $\Box$

Claim 5.19 *For making $G = (V, E)$ k-vertex-connected, $k \leq 4$, let $\mathcal{S}$ be a separating set in G with cardinality less than $k - 1$ and let $\delta = 2 \cdot a_k(\mathcal{S}, G) - \text{DEMAND}_k(\mathcal{S}, G)$. (1) If $\text{DEMAND}_k(V \setminus \mathcal{S}, G) \geq \delta + \text{DEMAND}_{k-|\mathcal{S}|}(V \setminus \mathcal{S}, G - \mathcal{S})$, then $\mathcal{S}$ is not massive. (2) If $\text{DEMAND}_k(V \setminus \mathcal{S}, G) > \delta + \text{DEMAND}_{k-|\mathcal{S}|}(V \setminus \mathcal{S}, G - \mathcal{S})$, then $\mathcal{S}$ is neither critical nor massive.*

Proof: Let $G' = G - \mathcal{S}$ and let $V' = V \setminus \mathcal{S}$. Let $\ell = |\mathcal{S}|$ and let $k' = k - \ell$. By Claim 5.18, if $\mathcal{S}$ is critical or massive, $\text{AUG}_{k'}(G') = \left\lceil \frac{\text{DEMAND}_{k'}(V', G')}{2} \right\rceil$. Thus $\text{SC}_k(\mathcal{S}, G) = a_k(\mathcal{S}, G) + \left\lceil \frac{\text{DEMAND}_{k'}(V', G')}{2} \right\rceil$. We also know that $\text{DEMAND}_k(V, G) \geq \text{DEMAND}_k(\mathcal{S}, G) + \text{DEMAND}_k(V', G) \geq \text{DEMAND}_k(\mathcal{S}, G) + \text{DEMAND}_{k'}(V', G')$. Thus if $\text{DEMAND}_k(V', G) \geq \delta + \text{DEMAND}_{k'}(V', G') = 2 \cdot a_k(\mathcal{S}, G) - \text{DEMAND}_k(\mathcal{S}, G) + \text{DEMAND}_{k'}(V', G')$, then $\left\lceil \frac{\text{DEMAND}_k(V, G)}{2} \right\rceil \geq \text{SC}_k(\mathcal{S}, G)$. If $\text{DEMAND}_k(V', G) > \delta + \text{DEMAND}_{k'}(V', G')$, then $\left\lceil \frac{\text{DEMAND}_k(V, G)}{2} \right\rceil > \text{SC}_k(\mathcal{S}, G)$. $\Box$

Claim 5.20 *For making a connected graph triconnected, (1) there is no massive separating 1-set, and (2) if there is a critical separating 1-set, then we can add an edge to reduce the demand of the graph by two and at the same time reduces the separation constraint of the graph by one.*

Proof: Let c be a cutpoint in $G = (V, E)$. Note that $a_3(\{c\}, G) \leq 1$. Let $V' = V \setminus \{c\}$ and let $G' = G - \{c\}$. By Claim 5.18, $\text{SC}_3(\{c\}, G) = a_3(\{c\}, G) + \left\lceil \frac{\text{DEMAND}_2(V', G')}{2} \right\rceil$. It is obviously true that $\text{DEMAND}_3(V, G) \geq \text{DEMAND}_3(\{c\}, G) + \text{DEMAND}_2(V', G')$.

Part (1): We first prove that $\{c\}$ cannot be massive.
Case 1: $a_3(\{c\}, G) = 1$. The degree of c is two and $\text{DEMAND}_3(\{c\}, G) = 1$. This also implies $\text{DD}(\{c\}, G) = 2$ and $\{c\}$ is a 2-block. If $\text{DEMAND}_2(V', G')$ is even, then $\text{SC}_3(\{c\}, G) \leq \left\lceil \frac{\text{DEMAND}_3(V, G)}{2} \right\rceil$. Thus $\{c\}$ cannot be massive. Assume that $\text{DEMAND}_2(V', G')$ is odd. Let $G' = G_1 \cup G_2$, where G_1 and G_2 are the two connected components in G'. Let V_i, $i \in \{1, 2\}$, be the set of vertices in G_i. Thus $\text{DEMAND}_2(V', G') = \sum_{i=1}^{2} \text{DEMAND}_2(V_i, G')$. Observe that

$\text{DEMAND}_2(V_i, G') \geq 2$, $i \in \{1, 2\}$. If $\text{DEMAND}_2(V', G')$ is odd, without loss of generality, assume that $\text{DEMAND}_2(V_1, G')$ is odd. Hence there are at least three 2-blocks with a positive demand for reaching biconnectivity. This implies $\text{DEMAND}_3(V_1, G) > \text{DEMAND}_2(V_1, G')$. By Claim 5.19, $\{c\}$ cannot be massive.

Case 2: $a_3(\{c\}, G) = 0$. Thus $\text{DEMAND}_3(\{c\}, G) = 0$. This implies $\text{SC}_3(\{c\}, G) \leq \left\lceil \frac{\text{DEMAND}_3(V, G)}{2} \right\rceil$. By Claim 5.19, $\{c\}$ cannot be massive.

Part (2): We now prove part (2) of the claim. Assume that $\{c\}$ is critical and there is another critical or massive separating ℓ-set $\mathcal{S}$, $\ell \leq 2$. We will discuss in the following two cases, the structure of G when $\{c\}$ is critical.

Case 1: $a_3(\{c\}, G) = 0$. Thus every component in $G - \{c\}$ is simple and there is no cutpoint other than c in G. If $\text{DD}(\{c\}, G) > 2$, then $\text{SC}_3(\{c\}, G) \geq 3$. If $\mathcal{S}$ is also critical, then $\text{SC}_3(\mathcal{S}, G) \geq 3$. However, in this case, there is a block with a positive demand whose degree does not reduce because of the removal of c. Thus $\text{DEMAND}_3(V', G') > \text{DEMAND}_2(V', G)$. By Claim 5.19, $\{c\}$ cannot be critical. Thus if $\mathcal{S}$ is critical, $\text{DD}(\{c\}, G) = 2$.

Case 2: $a_3(\{c\}, G) = 1$. Thus $\text{DD}(\{c\}, G) = 2$. We will prove that the dividing degree of any cutpoint $d \neq c$ is two and there is no critical or massive separating pair. Assume that $\text{DD}(\{d\}, G) > 2$, then there are two 2-blocks with positive demands whose degrees do not decrease because of the removal of d. Thus $\text{DEMAND}_3(V', G') \geq \text{DEMAND}_2(V', G) + 2$. By Claim 5.19, $\{c\}$ cannot be critical if $\text{DD}(\{d\}, G) > 2$. Note that $\text{SC}_3(\{c\}, G) \geq 3$. If there is a critical or massive separating pair, then its dividing degree is at least four. However, this implies $\text{DEMAND}_3(V', G') \geq \text{DEMAND}_2(V', G) + 2$. Thus there is no critical or massive separating pair.

From the above discussion on the structure of G, we know that we can find two 3-blocks $\mathcal{B}_1$ and $\mathcal{B}_2$ with positive demands and $\mathcal{B}_i$ is properly contained in a 2-block $\mathcal{Z}_i$ with a positive demand. Both $\mathcal{B}_i$ and $\mathcal{Z}_i$ contain c or are adjacent to c. Furthermore, every cutpoint in G separates a vertex in $\mathcal{B}_1$ and a vertex in $\mathcal{B}_2$. By the way the demand is computed, the demand of the graph is decreased by two by adding an edge between a demanding vertex in $\mathcal{B}_1$ and a demanding vertex in $\mathcal{B}_2$. The separation constraint of every critical separating set is also decreased by adding such an edge. □

Claim 5.20 states that in augmenting an undirected graph to reach triconnectivity, the separation constraints of separating 1-sets do not "reveal" any additional information in computing the smallest triconnectivity augmentation number. Thus one can derive an algorithm for finding a smallest triconnectivity augmentation without keeping track of separation constraints of separating 1-sets. This simplified approach is exactly the one used in [HR91] to construct a linear-time algorithm. However, it is possible that there is a massive separating ℓ-set to reach k-vertex-connectivity, for some $\ell < k - 1$. In Figure 3, we give a graph containing a massive separating 1-set for reaching four-connectivity.

Figure 3: Illustrating a graph containing a massive separating ℓ-set for reaching k-vertex-connectivity, $\ell < k-1$. Note that $\textsc{sc}_4(\{1\}, G) = 5$ and $\textsc{demand}_4(V, G) = 8$. Thus $\{1\}$ is a massive separating 1-set for reaching four-connectivity. We need to add five more edges to four-connect G. A smallest four-connectivity augmentation using exactly five edges is shown in G'.

5.5 Structure of Massive Separating Sets

The following claim is from [ET76, HR91, Hsu92].

Claim 5.21 ([ET76, HR91, Hsu92]) *If G is $(k-1)$-vertex-connected, $k \leq 4$, then the followings are true for making G k-vertex-connected.* (1) *There is at most one massive separating $(k-1)$-set.* (2) *There are at most two critical separating $(k-1)$-sets.* (3) *If there is a massive separating $(k-1)$-set, then there is no critical separating $(k-1)$-set.*

Claim 5.22 *Let $\mathcal{S}$ be a massive separating triplet for making a connected graph $G = (V, E)$ four-connected.* (1) *If $a_4(\mathcal{S}, G) = 0$, then $\textsc{sd}(\mathcal{S}, G) \geq 4$ and there are at least four 3-components in $G - \mathcal{S}$ which each contributes exactly one in computing $\textsc{demand}_k(V, G)$.* (2) *If $a_4(\mathcal{S}, G) > 0$, then G is triconnected, $\textsc{sd}(\mathcal{S}', G) = 2$ for every separating triplet $\mathcal{S}' \neq \mathcal{S}$, and there is no special block that is not in $\mathcal{S}$.* (3) *There is no other massive separating triplet.*

Proof: Let q_i be the number of i-components in $G - \mathcal{S}$.

Part (1): $\textsc{sc}_4(\mathcal{S}, G) = \sum_{i=1}^{3} q_i - 1$. Let $q_3 = q_3' + q_3''$ where q_3' is the number of 3-components which contributes exactly one in computing $\textsc{demand}_k(V, G)$ and q_3' is the number of other 3-components contributing at least two in computing $\textsc{demand}_k(V, G)$. By Claim 5.3, $\textsc{demand}_4(V, G) \geq q_3' + 2 \cdot q_3'' + 2 \cdot q_2 + 3 \cdot q_3$. In order for $\textsc{sc}_4(\mathcal{S}, G) > \left\lceil \frac{\textsc{demand}_4(V, G)}{2} \right\rceil$, $q_3' \geq 4$.

Part (2): $\textsc{sc}_4(\mathcal{S}, G) = a_4(\mathcal{S}, G) + \sum_{i=1}^{3} q_i - 1$. Note that $2 \cdot a_4(\mathcal{S}, G) - \textsc{demand}_4(\mathcal{S}, G) \leq 2$ and $q_3 \leq 3$. If $2 \cdot a_4(\mathcal{S}, G) - \textsc{demand}_4(\mathcal{S}, G) = 2$, then $q_3 = 2$ and $q_2 = 0$. Thus $\textsc{demand}_4(\mathcal{S}, G) \geq \textsc{demand}_4(\mathcal{S}, G) + q_3 + 2 \cdot q_2 + 3 \cdot q_1$. If $q_2 + q_1 > 0$, G is not triconnected, there is another separating triplet with separating degree more than two, or there is a special block not in $\mathcal{S}$, then $\frac{\text{DEMAND}_4(\mathcal{S},G)}{2} \geq a_4(\mathcal{S},G) + \sum_{i=1}^{3} q_i - 1$.

Part (3): From (2) and Claim 5.21, if $a_4(\mathcal{S},G) > 0$, then there is no other massive separating triplet. Assume that $a_4(\mathcal{S},G) = 0$. By (1), the separating degree of $\mathcal{S}$ is at least four. Thus by (2), there is no massive separating triplet $\mathcal{S}'$ with $a_4(\mathcal{S}',G) > 0$ and $\mathcal{S}' \neq \mathcal{S}$. □

Claim 5.23 *Let $\mathcal{S}$ be a massive separating ℓ-set, $\ell < 3$, for making a connected graph $G = (V,E)$ four-connected.* (1) $a_4(\mathcal{S},G) > 0$. (2) *There is no other massive separating h-set, $h \leq 3$.*

Proof:

Part (1) directly follows from Claim 5.19.(1).

Part (2): Assume that $\mathcal{S}'$ is a massive separating h-set, $h \leq 3$, and $\mathcal{S}' \neq \mathcal{S}$.
Case 1: $h = 3$. From Claim 5.22.(2), $a_4(\mathcal{S}',G) = 0$. From Claim 5.22.(1), $\text{SD}(\mathcal{S},G) \geq 5$. Since $a_4(\mathcal{S},G) > 0$, it is not possible that $\mathcal{S} \subset \mathcal{S}'$. This implies there is a $|\mathcal{S}|$-component in $G - \mathcal{S}$ whose contributes two less in computing $\text{DEMAND}_{4-|\mathcal{S}|}(V \setminus \mathcal{S}, G - \mathcal{S})$ than in computing $\text{DEMAND}_4(V,G)$. By Claim 5.19 and by the fact that $2 \cdot a_4(\mathcal{S},G) - \text{DEMAND}_4(\mathcal{S},G) \leq 2$, $\mathcal{S}$ cannot be massive.
Case 2: $h < 2$. If $\mathcal{S} \cap \mathcal{S}' \neq \emptyset$, then $2 \cdot a_4(\mathcal{S},G) - \text{DEMAND}_4(\mathcal{S},G) = 1$ and $2 \cdot a_4(\mathcal{S}',G) - \text{DEMAND}_4(\mathcal{S}',G) = 1$.

Let $x = \text{DEMAND}_{k-|\mathcal{S}|}(V \setminus \mathcal{S}, G - \mathcal{S})$. By Claim 5.18, $\text{SC}_4(\mathcal{S},G) = a_4(\mathcal{S},G) + \left\lceil \frac{x}{2} \right\rceil$. $\text{DEMAND}_4(V,G) \geq \text{DEMAND}_4(\mathcal{S},G) + x$. Assume that $2 \cdot a_4(\mathcal{S},G) - \text{DEMAND}_4(\mathcal{S},G) = 1$. If $\mathcal{S}'$ is massive, then it is either the case that there is a special block not containing in $\mathcal{S}$ or there is a $|\mathcal{S}'|$-component in $G - \mathcal{S}'$ who contributes more in computing $\text{DEMAND}_k(V,G)$ than in computing $\text{DEMAND}_{k-|\mathcal{S}|}(V \setminus \mathcal{S}, G - \mathcal{S})$. Thus $\text{DEMAND}_4(V,G) > \text{DEMAND}_4(\mathcal{S},G) + x$. As a result, $\text{SC}_4(\mathcal{S},G) \leq \frac{\text{DEMAND}_4(V,G)}{2}$.

If $2 \cdot a_4(\mathcal{S},G) - \text{DEMAND}_4(\mathcal{S},G) = 2$, then $\text{DD}(\mathcal{S},G) = 2$ and $\mathcal{S} \cap \mathcal{S}' = \emptyset$. If $\mathcal{S}'$ is massive, then it is either the case that x is four and $\text{DEMAND}_4(V,G) > \text{DEMAND}_4(\mathcal{S},G) + x$ or $\text{DEMAND}_4(V,G) > \text{DEMAND}_4(\mathcal{S},G) + x + 1$. Thus $\mathcal{S}$ is not massive. We have derived a contradiction. Hence the claim holds. □

Corollary 5.24 *Let G be a connected graph that is not triconnected. For making G four-connected, if there is a massive separating set, then we can add an edge to reduce the separation constraint of G by one.*

Proof: Let $\mathcal{S}$ be the massive separating set. Assume that $|\mathcal{S}| = 3$. By Claim 5.22, there are at least four 3-components in $G - \mathcal{S}$ and there is no other massive separating set. By adding an edge (u, v) such that u and v are in different 3-components in $G - \mathcal{S}$, we reduce the dividing degree of $\mathcal{S}$. Thus the separation constraint of the graph is reduced.

Assume that $|\mathcal{S}| < 3$. By Claim 5.23, $a_4(\mathcal{S}, G) > 0$ and there is no other massive separating set. If $a_4(\mathcal{S}, G) = \text{DEMAND}_4(\mathcal{S}, G)$, we add an edge (u, v) such that u is a demanding vertex in a special block in $\mathcal{S}$. By doing so, $a_4(\mathcal{S}, G \cup \{(u, v)\}) = a_4(\mathcal{S}, G) - 1$. Thus the separation constraint of the graph is reduced. If $a_4(\mathcal{S}, G) < \text{DEMAND}_4(\mathcal{S}, G)$, then we can find two special blocks $\mathcal{B}_1$ and $\mathcal{B}_2$ in $\mathcal{S}$ with $u_i \in \mathcal{B}_i$ and $(u_1, u_2) \notin G$. $a_4(\mathcal{S}, G \cup \{(u, v)\}) = a_4(\mathcal{S}, G) - 1$. Thus the separation constraint of the graph is reduced. □

5.6 Structure of Critical Separating Sets

Claim 5.25 *Let $\mathcal{S}$ be a separating set with cardinality less than three in $G = (V, E)$. For making G four-connected, if $\mathcal{S}$ is critical, then we can add an edge to reduce the separation constraint of every critical separating set by one.*

Proof: Let R be the set of separating sets in G whose cardinalities are less than three.
Case 1: $a_4(\mathcal{S}, G) \leq 1$. From Claims 5.10 and 5.11, $\mathcal{S}$ is the only separating set in R whose dividing degree might be greater than two. Let H be a connected component in $G - \mathcal{S}$. Furthermore, if there is a critical separating triplet, then its dividing degree is no more than four and it is the only non-trivial separating triplet in H. Thus the dividing degree of $\mathcal{S}$ is two. It is possible to add an edge (u_1, u_2) such that every separating set in R with dividing degree two is no longer a separating set in the resulting graph. Furthermore, by adding this edge, the dividing degree of every critical separating triplet is decreased by one. We can also choose u_i to be a demanding vertex in an x-block with a positive demand, for all $x \leq 4$.
Case 2: $a_4(\mathcal{S}, G) = 2$. Let $H_1, \ldots, H_r$ be connected components in $G - \mathcal{S}$. Since $\mathcal{S}$ is critical, by Claim 5.19, there cannot be any g-simple components, $g > 2$. There is at most one 2-simple component. Furthermore, if there is one, then all other components are 0-simple components. There are at most two 1-simple components. The cases when there is no 2-simple component is similar to Case 1. By the structure revealed in Claim 5.12, if there is a 2-component, we can also use the approach as the one used in Case 1. □

We now state a claim for the structure of a set separating sets. For convenience, a set of separating sets $\{S_1, \ldots, S_k\}$ has a *sample set* $\{Q_1, \ldots, Q_r\}$ if $\{S_1, \ldots, S_k\}$ can be disjointedly partitioned into r nonempty partitions $W_1, \ldots, W_r$, where $Q_i \subseteq R$ for all $R \in W_i$ and Q_i is either a special block or a separating set. In the above, if $Q_i \subseteq S_j$, then S_j is *sampled* by Q_i. A sample set with the minimum cardinality is a *minimum sample set.*

Claim 5.26 *Let $\{S_1, \ldots, S_k\}$ be a set of separating triplets. If the cardinality of its minimum sample set is more than two, then we can find three separating triplets in $\{S_1, \ldots, S_k\}$ such that the intersection of every two of them does not contain a separating set or a special block.*

Proof: We pick a set B_1 in the samples and S_1^* from $\{S_1, \ldots, S_k\}$ whose sample is B_1. We then remove B_1 from the set of samples and all sets in $\{S_1, \ldots, S_k\}$ that are sampled by B_1. Using a strategy that is similar to the above, we pick S_2^* and S_3^*. Then S_1^*, S_2^*, and S_3^* are the three sets that we want. □

In the following claim, we will prove that critical separating triplets in four-connecting a graph are "nicely" structured.

Claim 5.27 *For making a connected graph $G = (V, E)$ four-connected, let $\mathcal{Y}$ be the set of critical separating triplets in G. The size of a minimum sample set for $\mathcal{Y}$ is at most two.*

Proof: If $|\mathcal{Y}| \leq 2$, then the size of its minimum sample set is no more than two. Thus we assume that $|\mathcal{Y}| > 2$ and the minimum cardinality of a sample set is more than two. By Claim 5.26, there are three separating triplets $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ such that $\mathcal{S}_i \cap \mathcal{S}_j$ is neither a separating set nor a special block, for all $1 \leq i < j \leq 3$.

We now prove that $\textsc{demand}_4(V, G)$ must be greater than $2 \cdot \textsc{sc}_4(\mathcal{S}_w, G)$ for some $1 \leq w \leq 3$. Thus $\mathcal{S}_w$ is neither critical nor massive. Let $w_{i,j}$ be the number of j-components in $G - \mathcal{S}_i$, where $1 \leq j \leq 3$ and $1 \leq i \leq 3$.

By Corollary 5.17, for any $1 \leq i \leq 3$,

$$\textsc{sc}_4(\mathcal{S}_i, G) = a_4(\mathcal{S}_i, G) + \sum_{j=1}^{3} w_{i,j} - 1.$$

Without loss of generality, assume that r_1 and r_2 are two integers such that $a_4(\mathcal{S}_{r_1}, G) + w_{r_1,3} \leq a_4(\mathcal{S}_i, G) + w_{i,3}$ and $w_{r_2,1} + w_{r_2,2} \geq w_{i,1} + w_{i,2}$ for all $1 \leq i \leq 3$. By Claim 5.3, Corollary 5.4 and Claim 5.5,

$$\textsc{demand}_4(V, G) \geq \sum_{i=1}^{3} (\textsc{demand}_4(\mathcal{S}_i, G) + w_{i,3}) + 2 \cdot w_{1,2} + 3 \cdot w_{1,1} - \begin{cases} 6 & \text{if } w_{1,1} > 2; \\ 5 & \text{if } w_{1,1} = 1 \text{ and } w_{1,2} > 1; \\ 4 & \text{otherwise.} \end{cases}$$

We know that $w_{i,3} > 1$ and $2 \cdot a_4(\mathcal{S}_i, G) - \textsc{demand}_4(\mathcal{S}_i, G) \leq 2$, for all $1 \leq i \leq 3$. It is impossible that

$$\textsc{sc}_4(\mathcal{S}_{r_1}, G) \geq \left\lceil \frac{\textsc{demand}_4(V, G)}{2} \right\rceil.$$

We have derived a contradiction. Thus the claim holds. □

Claim 5.28 *For making a connected graph $G = (V, E)$ k-vertex-connected, $1 < k \leq 4$, let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two distinct critical separating $(k-1)$-sets. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are in the same ℓ-block $\ell < k$, and $\mathcal{S}_1 \cap \mathcal{S}_2$ is not an union of separating sets, then G is ℓ-vertex-connected.*

Proof: Let $\mathcal{I} = \mathcal{S}_1 \cap \mathcal{S}_2$. Note that $\text{SC}_k(\mathcal{S}_i, G) = a_4(\mathcal{S}_i, G) + \text{DD}(\mathcal{S}_i, G) - 1$. Let $q_{i,j}$ be the number of j-components in $G - \mathcal{S}_i$.

If there is a special block in $\mathcal{I}$, then every vertex in this special block is not adjacent to any vertex in $\mathcal{S}_1$ or $\mathcal{S}_2$. By Corollary 5.8, the degree of any subset of vertices in $\mathcal{I}$ is three and $\mathcal{I}$ contains no separating set. Hence $k = 4$, $\text{SD}(\mathcal{S}_1, G) = \text{SD}(\mathcal{S}_2, G) = 2$, and $2 \cdot a_4(\mathcal{S}_i, G) - \text{DEMAND}_4(\mathcal{S}_i, G) \leq 1$. There are $q_{1,3} + q_{2,3} - 1$ 3-components, $q_{1,2} + q_{2,2}$ 2-components, and $q_{1,1} + q_{2,1}$ 1-components in $G - (\mathcal{S}_1 \cup \mathcal{S}_2)$. Assume without loss of generality, $a_4(\mathcal{S}_1, G) + q_{1,2} \leq a_4(\mathcal{S}_2, G) + q_{2,2}$. By Claims 5.3, $\text{DEMAND}_k(V, G) \geq \text{DEMAND}_k(\mathcal{S}_1, G) + (\sum_{i=1}^{2} \text{SD}(\mathcal{S}_i, G)) - 1 + 2 \cdot (q_{1,2} + q_{2,2}) + 3 \cdot (q_{1,1} + q_{2,1})$. If G is not triconnected, then it is either (1) $q_{1,2} + q_{2,2} + q_{1,1} + q_{2,1} = 0$ and $\text{DEMAND}_k(V, G) > \text{DEMAND}_k(\mathcal{S}_1, G) + (\sum_{i=1}^{2} \text{SD}(\mathcal{S}_i, G)) - 1$ (i.e., there is a 3-component containing a separating 1-set or a separating 2-set), (2) $q_{1,1} + q_{2,1} > 0$, (3) $q_{1,2} > 0$ and $q_{2,2} > 0$, or (4) there is a special block in $\mathcal{S}_2 \setminus \mathcal{I}$. In all cases, $\mathcal{S}_1$ is less than half of the demand of G and thus is not critical. We have reached a contradiction. Hence the claim is true.

Assume that there is no special block in $\mathcal{I}$ and G is not ℓ-vertex-connected. By Claims 5.3 and 5.5, $\text{DEMAND}_k(V, G) \geq \sum_{i=1}^{2} (\text{DEMAND}_k(\mathcal{S}_i, G) + q_{i,3}) + 2 \cdot q_{1,2} + 3 \cdot q_{1,1} - 1$. Without loss of generality, assume that $\text{SC}_k(\mathcal{S}_1, G) \leq \text{SC}_k(\mathcal{S}_2, G)$. Hence $\mathcal{S}_1$ is less than half of the demand of G and thus is not critical. □

5.7 Reducing the Separation Constraint

Claim 5.29 *Let u_1 and u_2 be two vertices in a connected graph $G = (V, E)$ that are not four-connected and $(u_1, u_2) \notin E$. Let $\mathcal{S}$ be a separating triplet in G. Then $\text{SC}_4(\mathcal{S}, G \cup \{(u_1, u_2)\}) = \text{SC}_4(\mathcal{S}, G) - 1$ if (1) $\text{SC}_4(\mathcal{S}, G) > 2$, u_i is a demanding vertex in an x-block, for all $4 - \ell \leq x \leq 4$, and $\mathcal{S}$ separates u_1 and u_2, (2) u_1 and u_2 are demanding vertices in two distinct special blocks in $\mathcal{S}$, or (3) u_1 is a demanding vertex in $\mathcal{S}$, $u_2 \notin \mathcal{S}$, and $a_4(\mathcal{S}, G) = \text{DEMAND}_4(\mathcal{S}, G)$.*

Proof: This claim holds for conditions (2) and (3), since adding such an edge reduces $a_4(\mathcal{S}, G)$ by one. We now prove this claim holds for condition (1). Since $\ell = 3$, the number of connected components in $(G \cup \{(u_1, u_2)\}) - \mathcal{S}$ is one less than the number of connected components in $G - \mathcal{S}$ for condition (1). Thus this claim holds. □

Lemma 5.30 *Let G be a connected graph that is not triconnected. For making G four-connected, if there is a critical or massive separating set, we can find two vertices u and v in G such that the separation constraint of G is reduced by one by adding the edge (u, v).*

Proof: If there is a massive separating set, then by Claims 5.22 and 5.23 it is the only separating set that is massive and there is no critical separating set. We find u and v as described in the proof of Corollary 5.24.

If there is a critical separating set with cardinality less than three, we find u and v as described in the proof of Claim 5.25.

For the rest of the discussion, let ℓ be the vertex-connectivity of G. Assume that there is a critical separating triplet. If there is exactly one critical separating triplet $\mathcal{S}$, then we can find u and v such that they demanding vertices in x-blocks, for all $2 \leq x \leq 4$. Furthermore, u and v are separated by $\mathcal{S}$ and every separating ℓ-set in G.

If there are more than one critical separating triplet, we find a minimum sample set $\mathcal{K}$ whose cardinality less than three (Claim 5.27) by exhaustively enumeration. Let W_1 be an arbitrary set in $\mathcal{K}$. If W_1 is a special block, then W_1 is an union of exactly two critical separating sets whose separating degrees are two. By checking all possible configurations of a separating triplet with special blocks, in order to have two critical separating triplets the followings must be true: (1) W_1 is a degree-3 vertex that is a unique special block in each critical separating triplet, (2) there are exactly two critical separating triplets, and (3) G is triconnected. Thus we may assume that every set in $\mathcal{K}$ is a separating set by itself. Note that if $\mathcal{K}$ contains a separating x-set, $x < 3$, then by Claim **??** G is x-vertex-connected. Thus $\mathcal{K}$ cannot contain a separating 1-set and a separating 2-set at the same time. Let r be the smallest integer such that $\mathcal{K}$ contains a separating r-set. We can find u and v such that (1) they are demanding vertices in x-blocks, for all $2 \leq x \leq 4$, (2) they are in r-blocks with degree $r-1$, and (3) they are separated each time a set in $\mathcal{K}$ is removed. By adding the edge (u, v), the separating degree of every critical separating triplet is decreased by one. Thus the separation constraint of the graph is decreased by one. $\square$

6 Properties of Wheels

In the following lemma, we show that the separation constraint of a crossing separating set in a wheel cannot be massive.

Claim 6.1 *Let $\mathcal{W}$ be a wheel whose unit size is less than four. For reaching four-connectivity, any crossing separating set in $\mathcal{W}$ cannot be massive.*

Proof: Let $\mathcal{W}$ be a wheel in $G = (V, E)$. Let $\mathcal{S}$ be a crossing separating set in $\mathcal{W}$ and let x_i be the number of i-components in $G - \mathcal{S}$. By definition, $|\mathcal{S}| \in \{2, 3\}$.
Case 1: $|\mathcal{S}| = 3$. Then $x_3 = 2$. By Corollary 5.17, $\textsc{sc}_4(\mathcal{S}, G) \geq a_4(\mathcal{S}, G) + \sum_{i=1}^{3} x_i - 1$. By Claim 5.3, $\textsc{demand}_4(V, G) \geq \textsc{demand}_4(\mathcal{S}, G) + 3 \cdot x_1 + 2 \cdot x_2 + x_3$. In order for $\mathcal{S}$ to be massive, $2 \cdot \textsc{sc}_4(\mathcal{S}, G)$ must be greater than $\textsc{demand}_4(V, G) + 1$. Thus $2 \cdot a_4(\mathcal{S}, G) > \textsc{demand}_4(\mathcal{S}, G) + 1$. It is impossible for any crossing separating set to satisfy this condition.
Case 2: $|\mathcal{S}| = 2$. Then $x_2 = 2$ and $x_3 = 0$. By using an argument that is similar to the one used in Case 1, we can also derive the fact that $2 \cdot a_4(\mathcal{S}, G) > \textsc{demand}_4(\mathcal{S}, G) + 1$. Since

there is no edge between two distinct non-adjacent sides of a polygon, the above inequality does not hold.
This proves the claim. □

Note that if a separating set is massive, then its separation constraint is a lower bound for the smallest augmentation number. From this claim, we know that the separation constraints of crossing separating sets never "contribute any additional information" in lower bounding the smallest augmentation number. We will use this fact to speed up our algorithm for finding a smallest augmentation (Section 8).

We define a *wheel component* for a wheel $\mathcal{W}$ with unit size k as follows. It is either (1) a connected component in $G - \mathcal{W}$, or (2) a side with degree $k-1$. Intuitively, by adding an edge into each wheel component, we eliminate all separating k-sets represented by the wheel. Note that the wheel component for a wheel with unit size three is also defined in [Hsu92].

We define the *wheel constraint* of a wheel $\mathcal{W} = \mathcal{C} \cup \{W_0, W_1, \ldots, W_{q-1}\}$ for reaching k-vertex-connectivity to be $\left\lceil \frac{w(\mathcal{W})}{2} \right\rceil + a_k(\mathcal{C}, G)$ and is denoted as $\textsc{wc}_k(\mathcal{W}, G)$, where $w(\mathcal{W})$ is the number of wheel components in $\mathcal{W}$. The wheel constraint of a wheel is the smallest number of edges needed to add to eliminate all separating sets represented by the wheel.

Lemma 6.2 *Given a wheel $\mathcal{W}$ in G, we need to add at least $\textsc{wc}_k(\mathcal{W}, G)$ edges to k-vertex-connect G.* □

Note that a version of Lemma 6.2 is reported in [Hsu92] for the case of $k = 4$ and G is triconnected.

Lemma 6.3 *Let $\mathcal{W}$ be a wheel in $G = (V, E)$ with unit size three. For making G four-connected, (1) if $\textsc{wc}_4(\mathcal{W}, G) > \left\lceil \frac{\textsc{demand}_4(V, G)}{2} \right\rceil$, then G is triconnected, and (2) if $\textsc{wc}_4(\mathcal{W}, G) = \left\lceil \frac{\textsc{demand}_4(V, G)}{2} \right\rceil$, then G is biconnected and the separating degree of each separating pair is two.*

Proof: Assume that there are w wheel components in $\mathcal{W}$. Note that each wheel component contributes at least one in computing $\textsc{demand}_4(V, G)$ and $\textsc{demand}_4(\{a\}, G) = a_4(\{c\}, G)$. Thus $\textsc{demand}_4(V, G) \geq w + a_4(\{c\}, G)$ and $\textsc{wc}_4(\mathcal{W}, G) = a_4(\{c\}, G) + \left\lceil \frac{w}{2} \right\rceil$. The only case that the wheel constraint is greater than the ceiling of half of the demand of the graph is when the degree of the vertex c in the center is degree-3, in which case $a_4(\{c\}, G) = 1$. If G is not biconnected, then it is either the case that there is a cutpoint in a wheel component or the case that there is a 1-component not containing $\mathcal{W}$. Thus $\textsc{demand}_4(V, G) \geq w + 2 + a_4(\{c\}, G)$. Thus the wheel constraint is less than $\left\lceil \frac{\textsc{demand}_4(V, G)}{2} \right\rceil$. If there is a separating pair with separating degree more than two, then $\text{DEMAND}_4(V,G) \geq w + 3 + a_4(\{c\},G)$. Hence the wheel constraint is less than $\left\lceil \frac{\text{DEMAND}_4(V,G)}{2} \right\rceil$. □

Note that in [Hsu92], a *star wheel* is a wheel whose wheel constraint satisfies (1) of Lemma 6.3. The center of the star wheel is a degree-3 vertex.

7 Updating the Four-Block Graph

After adding the edge (u,v) to G, we show methods to obtain 4-$\text{BLK}(G \cup \{(u,v)\})$ by performing local operations on 4-$\text{BLK}(G)$.

In the discussion, we assume that the two endpoints of the added edge are demanding vertices of 4-blocks. Note that similar results for maintaining a structure that is analogous to our 4-block graph are reported in [KTDBC91] under the case of adding an arbitrary edge.

7.1 Merging of Blocks

The following claim identifies the set of h-blocks that will be merged into an h-block after adding an edge.

Claim 7.1 *Let u and v be demanding vertices in two h-blocks, $h > 1$, of G. All h-blocks $\mathcal{B}$ that are in the same $(h-1)$-block satisfying at least one of the following conditions merge into an h-block in $G \cup \{(u,v)\}$. (1) $\mathcal{B}$ contains u or v. (2) Let w be a vertex in $\mathcal{B}$. There is no separating $(h-1)$-set $\mathcal{S}$ such that w is separating from both u and v after removing $\mathcal{S}$.*

Proof: Let $\mathcal{H}$ be the set of all h-blocks that are merged into an h-block as specified in the claim. Observed that there are $h-1$ internally vertex-disjoint paths between every two vertices w_1 and w_2 in blocks of $\mathcal{H}$ before adding the edge (u,v). By adding the edge (u,v), w_1 and w_2 cannot be separated by removing any separating $(h-1)$-set. Thus vertices in h-blocks of $\mathcal{H}$ are in the same h-block after adding the edge (u,v). It is also easy to prove that there is no vertex x such that x is not in any block of $\mathcal{H}$, but x is in the same h-block with vertices in $\mathcal{H}$ after adding the edge (u,v). □

Note that a demanding vertex in an h-block must also be a demanding vertex in some x-block for all $x < h$. Thus if u and v are in 4-blocks, we might have to merge 1-blocks. For each x-block merged, we have to merge several $(x+1)$-blocks within it, for all $1 \leq x \leq 3$.

Updating the Demand

If the input graph is triconnected and we want to raise its vertex-connectivity to four, the conditions under which the addition of an edge reduces the demand of the graph by two are discussed in [Hsu92]. We give a claim for the general case of raising the vertex-connectivity by more than one.

Claim 7.2 *Let ℓ be the vertex-connectivity of a connected graph $G = (V, E)$ and let u_1 and u_2 be two vertices in G. Let $G' = G \cup \{(u_1, u_2)\}$, where $(u_1, u_2) \notin E$. Then* $\text{DEMAND}_k(V, G') = \text{DEMAND}_k(V, G) - 2$ *if* (1) $4 \geq k > \ell + 1$, (2) u_1 *and* u_2 *are in different* $(\ell+1)$*-blocks with degree* ℓ, *and* (3) *for any* $\ell < x \leq k$, u_i *is a demanding vertex in an* x*-block,* $i \in \{1, 2\}$.

Proof: Note that G is ℓ-vertex-connected. Thus V is an ℓ-block. By Claim 7.1, adding the edge (u_1, u_2) causes several $(\ell+1)$-blocks to be merged into an $(\ell+1)$-block. Let $\mathcal{B}$ the $(\ell+1)$-block created by merging the set of $(\ell+1)$-blocks $\{\mathcal{B}_1, \ldots, \mathcal{B}_q\}$. Given an x-block $\mathcal{B}_1$, an $(x+1)$-block $\mathcal{B}_2$, and a demanding vertex u in $\mathcal{B}_2$, u is a demanding vertex of $\mathcal{B}_1$ if $\mathcal{B}_2 \subseteq \mathcal{B}_2$. Without loss of generality, assume that $u_i \in \mathcal{B}_i$, $i \in \{1, 2\}$. Hence $\text{DEMAND}_k(\mathcal{B}_i, G') = \text{DEMAND}_k(\mathcal{B}_i, G) - 1$. The demand of any $(\ell+1)$-block not containing u_1 or u_2 stays the same.

By Corollary 4.7, $\text{DEMAND}_k(V, G') \geq \text{DEMAND}_k(V, G) - 2$. By the definition used to calculate the demand of the graph, $\text{DEMAND}_k(V, G') > \text{DEMAND}_k(V, G) - 2$ if and only if $\text{DEMAND}_k(\mathcal{B}, G') > \sum_{i=1}^{q} \text{DEMAND}_k(\mathcal{B}_i, G')$. This above condition also implies $\mathcal{B} \neq V$ and $k - \text{DEGREE}(\mathcal{B}, G) > (\sum_{i=1}^{q} \text{DEMAND}_k(\mathcal{B}_i, G)) - 2$. Since $\text{DEGREE}(\mathcal{B}_1, G) = \ell$ and $\text{DEGREE}(\mathcal{B}_q, G) = \ell$, $\sum_{i=1}^{q} \text{DEMAND}_k(\mathcal{B}_i, G) \geq 2 \cdot (k - \ell)$. By the fact that $k - \ell \geq 2$, $\sum_{i=1}^{q} \text{DEMAND}_k(\mathcal{B}_i, G) \geq 4$. Since $k \leq 4$ and $\text{DEGREE}(\mathcal{B}, G) \geq \ell$, $(\sum_{i=1}^{q} \text{DEMAND}_k(\mathcal{B}_i, G)) - 2 \geq k - \text{DEGREE}(\mathcal{B}, G)$. We have derived a contradition. This proves the claim. □

Corollary 7.3 *By adding an edge as described in the proof of Lemma 5.30, the demand of the graph is decreased by two if there is a critical separating set.*

Proof: If the edge e is added according to the proof of Claim 5.25, then the two endpoints of e are in different ℓ-blocks with degree $\ell - 1$ and the graph is $(\ell-1)$-vertex-connected. Thus by Claim 7.2 and Claim 5.28, the demand of the graph is decreased by two.

If e is added according to the proof of Claim 5.29, then it is either the case that G is triconnected or the two endpoints of e satisfy the conditions in Claim 7.2. However, we know that G is not triconnected. Thus by Claim 5.28, this corollary is true. □

7.2 Creation of Separating Sets

A separating set can be created by adding an edge (u, v) as described in the following way. Let $\mathcal{S}$ be a separating set in G whose separating degree is two. (Recall that the separating

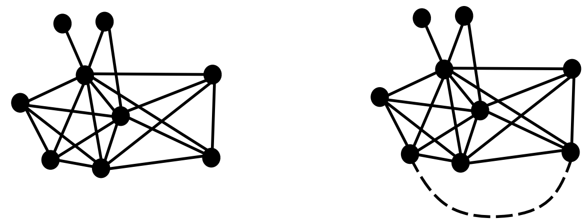

Figure 4: Illustrating the creation of a separating $(\ell+1)$-set from a separating ℓ-set by adding an edge. In G, {1, 2, 3} is a separating triplet. After adding the edge (4, 5), {1, 2, 3} is no longer a separating triplet. Two separating 4-sets {1, 2, 3, 4} and {1, 2, 3, 5} are created in G'.

degree is the number of different $|\mathcal{S}|$-components in $G-\mathcal{S}$.) Then $\mathcal{S}\cup\{u\}$ and $\mathcal{S}\cup\{v\}$ are two separating sets if u and v are in two distinct non-trivial $|\mathcal{S}|$-components. (Note that a trivial component is a connected component with exactly one vertex.) An example is given in Figure 4.

Let u and v be two vertices in $G=(V,E)$. A set of separating ℓ-sets $\mathcal{X}$ in the same ℓ-block is *co-linear* with respect to two vertices u and v, if every separating set in $\mathcal{X}$ separates u and v. Let $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ be three separating sets that are co-linear with respect to u and v. Then $\mathcal{S}_2$ is *in-between* $\mathcal{S}_1$ and $\mathcal{S}_3$ if there is a vertex w in $\mathcal{S}_1$ and a vertex w' in $\mathcal{S}_3$ such that $\mathcal{S}_2$ separates w and w'.

Claim 7.4 *Given two vertices u and v in G and a set of co-linear separating ℓ-sets $\mathcal{M}$, $|\mathcal{M}|>2$, we can uniquely order separating ℓ-sets in $\mathcal{M}$ as $\mathcal{S}_1,\ldots,\mathcal{S}_{|\mathcal{M}|}$, such that $\mathcal{S}_i$ is in-between $\mathcal{S}_{i-1}$ and $\mathcal{S}_{i+1}$, for all $2\le i\le|\mathcal{M}|-1$.* □

Note that the two separating sets created by adding an edge as specified in the paragraphs before Claim 7.4 are co-linear. By the introduction of these two co-linear separating sets, two sets of co-linear separating sets are concatenated into a set of co-linear separating sets. An example is illustrated in Figure 5.

We now identify the condition under which a set of co-linear separating sets is created by adding an edge in addition to the case of concatenating two existing sets of co-linear separating sets.

Claim 7.5 *Let u_1 and v_1 be two vertices in $G=(V,E)$ such that $(u_1,v_1)\notin E$. Given a set of co-linear separating ℓ-sets $\mathcal{M}$ with respect to u_1 and v_1, then the following two sets are co-linear in $G\cup\{(u_1,v_1)\}$. (1) The set of separating $(\ell+1)$-sets created by adding u_1 to each*

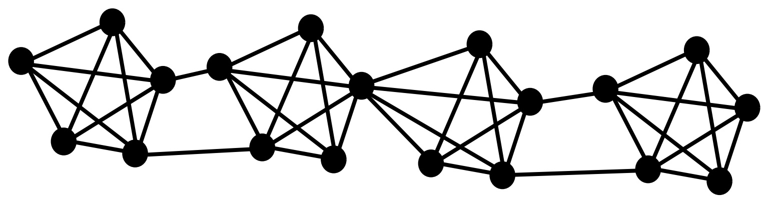

Figure 5: Illustrating sets of co-linear separating sets and the concatenating of two sets of co-linear separating sets by adding an edge. The sets {{1, 2}, {3, 4}} and {{9, 10}, {11, 12}} are both sets of co-linear separating sets with respect to vertices 13 and 14. After adding the edge (6, 7), {{1, 2}, {3, 4}, {5, 6}, {5, 7}, {9, 10}, {11, 12}} is a set of co-linear separating set with respect to vertices 13 and 14.

separating ℓ-set in $\mathcal{M}$. (2) *The set of separating $(\ell+1)$-sets created by adding v_1 to each separating ℓ-set in $\mathcal{M}$.* $\square$

Thus we can create a set of co-linear separating sets with a non-empty common intersection by adding an edge. Two sets of co-linear separating sets $\mathcal{M}_1$ and $\mathcal{M}_2$ created from the set of co-linear separating sets $\mathcal{M}$ by adding the edge (u_1, v_1) as described in Claim 7.5 can be denoted as $\mathcal{M} \oplus (u_1, v_1)$. Note that if another edge (u_2, v_2) is added and $\mathcal{M}$ is also co-linear with respect to u_2 and v_2 in G, then (up to) four sets of co-linear separating sets are created in $G \cup \{(u_1, v_1), (u_2, v_2)\}$ and can be denoted as $\mathcal{M} \oplus (u_1, v_1) \oplus (u_2, v_2)$. We will assume that each pair (u_i, v_i) is an ordered pair such that $\mathcal{M}$ is co-linear with respect to u_i and v_j in G, for all i and j. Note that for finding a smallest augmentation, there is no need to maintain co-linear sets. They can be recognized on the spot when we traverse the block graph from the block containing one endpoint to the block containing the other endpoint of an added edge.

7.3 Creation of Wheels

By further adding edges, a set of co-linear separating sets becomes a wheel. Before the statement of a claim to characterize the above, we give the following definition. Given a set of co-linear ℓ-separating sets $\{\mathcal{S}_1, \ldots, \mathcal{S}_p\}$, the *center* $\mathcal{C}$ is a subset of vertices in $\cap_{i=1}^{p} \mathcal{S}_i$ such that (1) $|\textsc{neighbor}(\mathcal{C}, G) \cap H| \geq |\mathcal{C}|$, and (2) every vertex in $\mathcal{C}$ is adjacent to (in the original graph) a vertex in H, for every H that is a connected component in $G - (\mathcal{S}_i \cup \mathcal{S}_{i+1})$, $1 \leq i < p$.

The following claim identifies the conditions under which a new wheel is created by adding an edge. It can be easily verified.

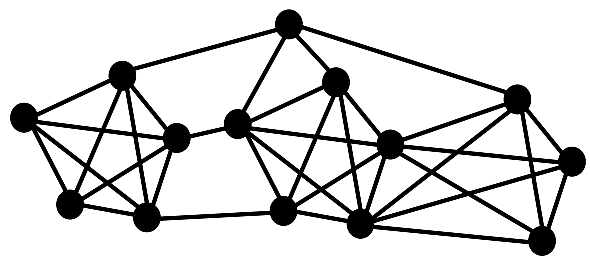

Figure 6: Illustrating the creation of a wheel by adding an edge. The set {{9, 1, 2}, {9, 3, 4}, {9, 5, 6}} is co-linear with respect to vertices 7 and 8, and its center is {9}. By adding the edge (7, 8), a wheel with center {9} and sides {1, 2}, {3, 4}, and {5, 6} is created.

Claim 7.6 *Let $G = (V, E)$ be a graph with sets of co-linear separating sets $\mathcal{M} \oplus (u_1, v_1) \oplus \cdots \oplus (u_{p-1}, v_{p-1})$ and let u_p and v_p be two vertices in G. Let $\mathcal{M}$ be co-linear with respect to u_i and v_j, for all $i \neq j$, and let $\mathcal{M}$ contain h separating sets. Let S_i be the ith separating set in the unique ordering of members in $\mathcal{M}$ as illustrated in Claim 7.4 and let $\mathcal{C}$ be the center of $\mathcal{M}$. If the cardinality of every separating set in $\mathcal{M}$ is $p + |\mathcal{C}|$, then $G \cup \{(u_p, v_p)\}$ contains a wheel with center $\mathcal{C}$ and sides $W_1, \ldots, W_{h+2}$ where (1) $W_i = \mathcal{S}_i$, $1 \leq i \leq h$, (2) $W_{h+1} = \{u_1, \ldots, u_{p-1}, u_p\}$, and (3) $W_{h+2} = \{v_1, \ldots, v_{p-1}, v_p\}$.* □

Note that the special case of creating a polygon by adding an edge (u_1, v_1) is given in [HR91]. Note also that the number of sides in a wheel can increase if the underlying set of co-linear separating sets has added new members. An example is illustrated in Figure 6 for the creation of a wheel by adding an edge. Note that for reaching four-connectivity, we only deal with wheels whose unit sizes are two or three. If we always add edges whose two endpoints are demanding vertices of 4-blocks, then we can create a set of co-linear separating triplets whose vertex in the center is degree more than three. However, a set of co-linear separating triplets with a center vertex whose degree is less than four could exist in the original input graph.

7.4 Splitting of Wheels

The following claim states the modifications made on wheels after adding an edge. Before the statement of this claim, we give some definitions. Let $\mathcal{W}$ be a wheel in G with the center $\mathcal{C}$ and sides $W_1, \ldots, W_q$. Let k be the unit size of $\mathcal{W}$. Given a side W_i, $\textsc{next}(W_i) = W_{(i+1) \bmod q}$ and $\textsc{prev}(W_i) = W_{(q+i-1) \bmod q}$. The two sides of a vertex u, $\mathrm{Q}_1(u)$ and $\mathrm{Q}_2(u)$, are W_i and $W_{(i+1) \bmod q}$, respectively, if u is separated from vertices in $W_{(i+2) \bmod q}$ by removing $\mathcal{C} \cup W_i \cup W_{(i+1) \bmod q}$. If $u \in W_i$, then the two sides of u, $\mathrm{Q}_1(u)$ and $\mathrm{Q}_2(u)$, are $W_{(q+i-1) \bmod q}$

and $W_{(i+1) \bmod q}$, respectively. Let u and v be two vertices in G. If u, v, and $\mathcal{W}$ are in the same k-block, then either $\mathcal{C} \cup \mathrm{Q}_1(u) \cup \mathrm{Q}_2(u) = \mathcal{C} \cup \mathrm{Q}_1(v) \cup \mathrm{Q}_2(v)$ or they are co-linear with respect to u and v.

Claim 7.7 *Let u and v be two vertices in G and let $\mathcal{W}$ be a wheel in G with the center $\mathcal{C}$ and sides $W_1, \ldots, W_q$. If the two sides of u and the two sides of v are not the same, then $\mathcal{W}$ is split into two wheels $\mathcal{W}_1$ and $\mathcal{W}_2$ in $G \cup \{(u,v)\}$, where* (1) *$\mathcal{C}$ is the center for both wheels,* (2) *the sides of $\mathcal{W}_1$ are $\mathrm{Q}_2(u), \textsc{next}(\mathrm{Q}_2(u)), \ldots, \textsc{prev}(\mathrm{Q}_1(v)), \mathrm{Q}_1(v)$, and* (3) *the sides of $\mathcal{W}_2$ are $\mathrm{Q}_2(v), \textsc{next}(\mathrm{Q}_2(v)), \ldots, \textsc{prev}(\mathrm{Q}_1(u)), \mathrm{Q}_1(u)$.*

Proof: Similar to a proof for updating polygons in triconnectivity augmentation [HR91]. □

The operation defined in Claim 7.7 is called *split*.

7.5 Updating the Structure

When an edge is added between two vertices that are demanding vertices, we apply the following updating operations to maintain the 4-block graph structure. Let u and v be the two endpoints of the new edge. If u and v are in different connected components, then the updating operations are trivial. Assume that they are in the same connected component. Using the 4-block graph, we can decide the largest integer i such that u and v are both in an i-block $\mathcal{B}_i$. Let P be the path in the $(i+1)$-block tree for $\mathcal{B}_i$ between the $(i+1)$-block containing u and the $(i+1)$-block containing v. We update the $(i+1)$-block tree for $\mathcal{B}_i$ by traversing the path P. Let $\mathcal{B}_{i+1}$ be a non-trivial $(i+1)$-block encountered when we traverse P. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the two separating i-sets in P that are both adjacent to $\mathcal{B}_{i+1}$. Let $\mathcal{B}'_{j,i+2}$ be an $(i+2)$-block containing $\mathcal{S}_j$, $j \in \{1,2\}$. We apply updating operations on the $(i+2)$-block tree for $\mathcal{B}_{i+1}$ as there is an edge added between a demanding vertex in $\mathcal{B}'_{1,i+2}$ and a demanding vertex in $\mathcal{B}'_{2,i+2}$. If we encounter a wheel in P, then we have to apply a split operation as described in Section 7.4. The above updating operation is applied recursively until we have reached the 4-block tree structure for each 3-blocks. The updating operations required to perform in each level of block trees are discussed in the above subsections.

8 An Algorithm for Four-Connectivity Augmentation

In this section, we first give a lower bound of the smallest four-connectivity augmentation number. Based on this lower bound, we give our algorithm for finding a smallest four-connectivity augmentation.

8.1 A Simple Lower Bound for the Augmentation Number

Given $G = (V, E)$, the *demand constraint* for making G k-vertex-connected is $\left\lceil \frac{\text{DEMAND}_k(V,G)}{2} \right\rceil$. The *wheel constraint* of G is the largest wheel constraint among all wheels with unit size less than k. Recall that the separation constraint of G is the largest separation constraint among all separating sets with cardinality less than k. We now give a theorem stating a lower bound based on the demand constraint, the separation constraint, and the wheel constraint of G. Let

$$\text{LOW}_k(G) = \max\{\left\lceil \frac{\text{DEMAND}_k(V,G)}{2} \right\rceil, \max_{\forall \mathcal{S}}\{\text{SC}_k(\mathcal{S}, G)\}, \max_{\forall \mathcal{W}}\{\text{WC}_k(\mathcal{W}, G)\}\},$$

where $\mathcal{S}$ is a separating set with cardinality less than k, and $\mathcal{W}$ is a wheel with unit size less than k in G.

Theorem 8.1 $\text{AUG}_k(G) \geq \text{LOW}_k(G)$.

Proof: By Lemmas 4.8, 5.14, and 6.2. □

Note that $\text{AUG}_4(G) = \text{LOW}_4(G)$ when G is triconnected [Hsu92]. Note also that $\text{AUG}_k(G) = \text{LOW}_k(G)$ for $k \leq 3$ [HR91, HR93].

8.2 The Algorithm

We now state an algorithm for finding a smallest four-connectivity augmentation on an undirected graph. Our algorithm is based on the following approach. Using the lower bound on the number of edges needed (Theorem 8.1) as a guideline, we make sure that each time we add an edge, this lower bound is decreased by one. We keep adding an edge until the graph is triconnected. Let $G = (V, E)$ be the resulting triconnected graph. We know that the lower bound given in Theorem 8.1 equals to $\text{AUG}_4(G)$ if G is triconnected. Thus we can apply the algorithm given in [Hsu92] to wrap up the computation. By doing this, we guarantee that the number of edges added is minimum in four-connecting the original input graph.

In finding a proper edge to add before the graph becomes triconnected, we use information available in its 4-block graph instead of getting the information directly form the current graph. Since the 4-block graph is a forest and we can update the 4-block graph fairly easily when an edge is added using properties given in Section 7, we can have a polynomial time implementation.

We describe our algorithm in Algorithm 1, prove its correctness, and analyze its complexity.

```
graph function aug0to4(graph G); {* The input graph G contains at least five vertices. *}
if G has exactly five vertices, then
return a complete graph with five vertices;
let T be 4-BLK(G);
let s be the separation constraint of G;
let d be the demand constraint of G;
while G is not triconnected do
  if G is not connected, then
    use Claim 5.16 to find a pair of vertices u and v
1. else if d > 1 and s > d then
    if there is a massive separating set with cardinality less than three then
      use Claim 5.23 to find a pair of vertices u and v
    else if there is a massive separating triplet then
      use Claim 5.22 to find a pair of vertices u and v
    fi fi s := s − 1
2. else if d > 1 and s = d then
    else if there is a critical separating set with cardinality less than three then
      use Claim 5.25 to find a pair of vertices u and v
    else if there is a critical separating triplet then
      use Claim 5.27 to find a pair of vertices u and v
    fi fi s := s − 1; d := d − 1
3. else find two vertices u and v using Claim 7.2; d := d − 1
  fi fi fi
4. G := G ∪ {(u, v)}; update T
end while;
return aug3to4(G) {* Function aug3to4 finds a smallest four-connectivity augmentation
for a triconnected G [Hsu92]. *}
end aug0to4;
```

Algorithm 1: An algorithm for finding a smallest set of edges whose addition four-connects an undirected graph with at least five vertices.

$\boxed{\textbf{Correctness}}$

We now prove that algorithm `aug0to4` finds a correct solution.

Lemma 8.2 *Let u and v be the two vertices found in step 4 of algorithm* `aug0to4`. *If G is not triconnected, then* $\text{LOW}_4(G \cup \{(u,v)\}) = \text{LOW}_4(G) - 1$.

Proof: Note that the value of $\text{LOW}_4(G)$ depends upon the values of three constraints. If G is not connected, then by Claim 5.16, we can reduce $\text{LOW}_4(G)$ by one by adding an edge. A constraint *dominates* $\text{LOW}_4(G)$ if the value of that constraint is equal to $\text{LOW}_4(G)$. From Lemma 6.3, if G is not biconnected, then the wheel constraint does not dominate $\text{LOW}_4(G)$. It is also easy to see that if G is biconnected, but not triconnected, and the wheel constraint of a wheel is equal to the demand constraint, then reducing the demand constraint implies reducing the wheel constraint. Thus we only have to consider the following two cases.
Case 1: The demand constraint dominates the lower bound, but the separation constraint does not. Algorithm `aug0to4` finds u and v in step 3. By Claim 7.2, we know that the demand is reduced by two in the resulting graph. Thus the demand constraint is reduced by one.
Case 2: The separation constraint dominates. Our algorithm finds a pair of vertices in steps 1 and 2. Note that by Lemma 5.30, we can guarantee that the separation constraint is decreased by one by adding an edge. In the case of both the separation constraint and the demand constraint dominate the lower bound (i.e., there is a critical separating set), Corollary 7.3 guarantees that the demand constraint is reduced by one.

From the discussion in Cases 1 and 2, the lemma holds. $\square$

Theorem 8.3 *Algorithm* `aug0to4` *finds a smallest four-connectivity augmentation for G and* $\text{AUG}_4(G) = \text{LOW}_4(G)$.

Proof: By Theorem 8.1 and Lemma 8.2. $\square$

We note that algorithm `aug0to4` can be modified to find a smallest k-vertex-connectivity augmentation, for any $k \leq 3$, if we know the followings. (1) An algorithm to construct the k-block graph. (2) A formula to compute the smallest ℓ-vertex-connectivity augmentation number, for all $\ell < k$. This is needed for finding critical and massive separating sets. (3) An algorithm to find a smallest k-vertex-connectivity augmentation for a $(k-1)$-vertex-connected graph. This is needed to wrap up the whole computation as algorithm `aug3to4` does in algorithm `aug0to4`. We remark that we know all of the above for any $k \leq 3$.

$\boxed{\textbf{Complexity}}$

Let n and m be the numbers of vertices and edges in G. To implement algorithm `aug0to4`, we are required to perform the following computations. (1) The algorithm must maintain and update the 4-block structure once a new edge is added. (2) The identification of critical and massive separating sets under the conditions that new separating sets may be created and the separation constraint of an existing separating set may decrease. It takes $O(n \cdot \alpha(n,n)+m)$ time to construct the 4-block graph using routines in [HT73, KTDBC91]. In this structure, we can retrieve all needed information for our augmentation algorithm. Methods for updating the 4-block graph are discussed in Section 7 and are also reported in [KTDBC91]. They can be implemented using the standard dynamic tree manipulation operations as used in various vertex-connectivity augmentation algorithms [HR91, Hsu92, RG77]. Each operation takes $O(\log n)$ time. Note that $\text{AUG}_4(G) = O(n)$. Thus the overall time complexity for (1) during the entire execution of the algorithm is $O(n \cdot \log n)$.

The total size of the 4-block graph is $O(n)$ [KTDBC91]. To implement (2), note that we can dynamically maintain the separation constraints of separating triplets under the adding of edges using data structures similar to the ones used in various augmentation algorithms [RG77]. Note also that if there is a massive separating set, then it is the only separating set that is massive. We add edges to balance the graph. Once it becomes balanced, then it stays balanced. If there is a critical separating set, then its separation constraint is decreased each time an edge is added.

By Claims 5.23 and 5.25, the graph has the following special structure if there is a critical separating set with cardinality less than three. The 2-block graph is a star with a possible degenerated case of being a path. The 3-block graph for every but one 2-block is a path. The only 3-block tree that might not be a path is either a path or a star. If the 2-block graph is a star that is not a path, then all 3-block trees are paths. Let $\mathcal{S}$ be the separating set with cardinality less than three with separating degree more than two. $\mathcal{S}$ is an empty set if there is no such separating set. If this is the case, then we add an edge where their endpoints are demanding vertices of degree-0 or degree-1 2-blocks. The two endpoints are also demanding vertices of 3-blocks and are separated by $\mathcal{S}$ if $\mathcal{S} \neq \emptyset$. By doing this, the demand of the the graph is decreased by two and the added edge satisfy Claim 5.25 if there is a critical separating set with cardinality less than three. Hence we have the following lemma.

Lemma 8.4 *Algorithm* `aug0to4` *can be implemented to run in* $O(n \cdot \log n + m)$ *time.* □

Lemma 8.5 *The number* $\text{AUG}_4(G)$ *can be computed in* $O(n \cdot \alpha(n,n)+m)$ *time.*

Proof: The 4-block graph can be constructed in $O(n \cdot \alpha(n,n)+m)$ time. Note that the demand constraint can be computed in $O(n)$ time once we have the 4-block graph. The wheel constraints of all wheels can also be computed in $O(n)$ time since there are only $O(n)$ wheels. The separation constraints of separating triplets can be computed in $O(n)$ time using the 4-block graph, since there are $O(n)$ separating triplets that are regular and not crossing

(Claim 6.1). To compute the separation constraints of separating pairs, we need to compute $\textsc{aug}_2(G - \mathcal{S}_2)$ for all separating pairs $\mathcal{S}_2$ that are not crossing (where there are $O(n)$ of them) in G (Claim 6.1). Note that $\textsc{aug}_2(G - \mathcal{S}_2) = \max\{d_1 - 1, \left\lceil \frac{\textsc{demand}_2(V \setminus \mathcal{S}_2, G - \mathcal{S}_2)}{2} \right\rceil\}$, where d_1 is the maximum number of connected components obtained in $G - \mathcal{S}_2$ by removing a cutpoint. This value can be computed in $O(n)$ time once the 2-block graph is computed. The separation constraints of separating 1-sets, where there are $O(n)$ of them, can be computed in $O(n)$ time using an approach that is similar to the one used in computing the separation constraints of separating pairs. $\square$

9 Concluding Remarks

We have shown an $O(n \cdot \log n + m)$-time algorithm for finding a smallest four-connectivity augmentation where n and m are the number of vertices and edges in the input graph, respectively. We also have shown a formula to compute th smallest four-connectivity augmentation number in $O(n \cdot \alpha(n,n) + m)$ time, where α is the inverse of the Ackermann function.

Our paper not only has answered the algorithmic aspect of the vertex-connectivity augmentation problem, but also has studied several useful properties about the structure of a graph that is not four-connected, e.g., the separating ℓ-sets and $(\ell+1)$-blocks, for all $1 \leq \ell < 4$. Note that the separating sets with cardinality less than four can have non-trivial intersections. The blocks can also have non-trivial intersections. We have shown the evolution of this structure when an edge is added to optimally increase the vertex-connectivity of a graph.

In developing our algorithm for increasing the vertex-connectivity of an undirected graph to four, we have established theorems that might be useful in answering questions arising from solving the fundamental problem of raising the vertex-connectivity of an undirected graph by an arbitrary value.

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