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PERFECT GRAPHS*

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Abstract

The class of perfect graphs is tremendously rich in their combinatorial structures. Research in this area involves techniques drawn from polyhedral combinatorics, graph theory and computational complexity. Several well–known optimization problems such as the maximum clique, maximum independent set and minimum coloring problems are NP–hard on general graphs but solvable in polynomial time on perfect graphs. Many interesting classes of graphs encountered in theoretical as well as applied research turn out to be perfect. The duality relationships embedded in the definition of perfect graphs play an important role in their algorithmic and graph theoretic properties. The strong perfect graph conjecture of Berge, probably the most famous open problem in graph theory, has eluded researchers for almost thirty years. Research efforts surrounding this conjecture have yielded many deep and intriguing results which have applications beyond perfect graphs. Much of this work would not become obsolete even if the conjecture were proved. In this survey, we shall summarize the properties of perfect graphs, discuss the relationships among these properties, classify techniques developed for solving these problems and list existing algorithmic results and open problems.

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PERFECT GRAPHS

1. Introduction

The class of perfect graphs was discovered by Claude Berge. Motivated by Shannon's work on "zero error capacity of a noisy channel", Berge made the famous ***strong perfect graph conjecture*** (SPGC) in 1960. This conjecture has stimulated researchers for thirty years to strive for its resolution. Today, the conjecture remains open. However, the intensive research efforts have produced many interesting ideas and results. Much of this work would not become obsolete even if the SPGC were proved. In this survey, we shall discuss problems generated from the study of perfect graphs and techniques developed for solving these problems. Since there have been many books and review papers covering different aspects of perfect graphs (see the books of Golumbic [1980] and Berge and Chvátal [1984]), we will emphasize more on recent results, in particular, the algorithmic aspects of perfect graphs, which have applications beyond perfect graphs.

We consider graphs that are undirected with no loops or multiple edges. A graph G is denoted by a pair (V,E), where V (or V(G)) denotes the finite vertex set of G, and E (or E(G)) denotes a set of edges connecting vertices of G. A subgraph H of G is an ***induced subgraph*** if E(H) consists of all edges in G both of their end vertices are in V(H). A subset $P \subseteq V(G)$ is an ***independent*** (or ***stable***) set in G if no two vertices in P are adjacent in G. A subset $Q \subseteq V(G)$ is a ***clique*** if every two vertices of Q are adjacent in G. A collection $\mathscr{C}$ of cliques is said to be a ***clique cover*** of G if the union of cliques in $\mathscr{C}$ is V(G). A ***coloring*** of G is an assignment of colors to vertices of G so that no two adjacent vertices receive the same color. Such a coloring gives rise to a collection of independent sets whose union is V(G). Define the complement $\overline{G}$ of G to be the graph with vertex set V(G) such that two vertices in $\overline{G}$ are adjacent if and only they are not adjacent in G. Thus, a clique in G becomes an independent set in $\overline{G}$ and a clique cover of G becomes a stable set cover of $\overline{G}$. Finally, a class $\mathscr{C}$ of graphs is said to be a ***hereditary class*** if $G \in \mathscr{C}$ implies that every induced subgraph of G is also in $\mathscr{C}$.

There are four important parameters associated with a graph G, which play important roles in many graph optimization problems:

$\alpha(G)$: the maximum size of an independent set in G.

$\theta(G)$: the least number of cliques in a clique cover of G.

$\omega(G)$: the maximum size of a clique in G.

$\gamma(G)$: the least number of colors in a coloring of G.

A graph G is ***perfect*** if $\alpha(H) = \theta(H)$ for every induced subgraph H of G. Historically, such a graph was defined to be ***α–perfect*** and there was a separate definition of "***γ–perfect***": $\omega(H) = \gamma(H)$ for every induced subgraph H of G. Two obvious classes of imperfect graphs can be defined as follows. Let the ***length*** of a cycle C be the number of edges in C and a ***chord*** of C be an edge connecting two non–consecutive vertices in C. Define an induced chordless cycle on n vertices with $n \geq 4$ to be a ***hole*** (denoted by C_n) in G and its complement to be an ***antihole*** (denoted by $\overline{C}_n$). It is easy to see that for odd n, C_n and $\overline{C}_n$ are imperfect. In the early 60's, Berge made two conjectures (Berge [1961,1962]). The first conjecture, later referred to as the strong perfect graph conjecture, states that a graph G is perfect iff G contains no odd holes or odd antiholes. The second conjecture, which was proved by Lovász [1972a] and referred to as the Perfect Graph Theorem, states that a graph is α–perfect iff it is γ–perfect, or equivalently, G is perfect iff $\overline{G}$ is perfect.

Studies on perfect graphs range from their graph theoretic properties to their algorithmic implications. Hereditary graphs can often be characterized by forbidden induced subgraphs. The SPGC suggests a simple forbidden structures for perfect graphs, which has been verified for many special cases. The problems for finding the above four parameters are NP–hard on general graphs. However, they are polynomially solvable for perfect graphs. The first few classes of graphs known to be perfect are triangulated graphs, comparability graphs and i–triangulated graphs. The list of these special classes of perfect graphs grows rapidly and efficient polynomial algorithms for the four parameters were known for most of them. In 1980, Grötschel, Lovász and Schrijver [1980,1984] presented polynomial algorithms for finding these parameters on general perfect graphs based on the separation concept of the ellipsoid algorithm. However, these algorithms are unlikely to be practical and efficient combinatorial algorithms exploiting the graphical structure of perfect graphs are still being pursued actively.

Research on perfect graphs has become very active in recent years. Many new tools have been developed for attacking their recognition and optimization problems. We shall divide our discussion into the following four sections. In the next section, we describe the properties of perfect graphs as well as imperfect graphs. Section 3 investigates the relationships among the rapidly growing collection of special classes of perfect graphs. Recognition and optimization problems on perfect graphs and various approaches towards their resolution are presented in Section 4. Finally, in Section 5, we consider related problems and applicatioins.

2. Properties of Perfect Graphs

Much of the research on perfect graphs has been motivated by the two conjectures of Berge. After the weak conjecture was solved by Lovász in 1972, a large body of work has been directed to imperfect graphs, especially those that are ***minimal imperfect.*** This is the class of imperfect graphs each of whose proper induced subgraph is perfect. It is easy to see that the class of odd holes and the class of odd antiholes are minimal imperfect. Berge's SPGC states that these are the only minimal imperfect graphs. This conjecture has been tackled through techniques from both graph theory and polyhedral combinatorics. Recently, a new property of perfect graphs stronger than the Perfect Graph Theorem but weaker than the SPGC was conjectured by Chvátal [1984] and verified by Reed [1987]. This property is based on the consideration of the P_4–structure in perfect graphs, where a P_4 is an induced chordless path with four vertices.

Our discussion is divided into four sections: (2.1) the Perfect Graph Theorem; (2.2) minimal imperfect graphs; (2.3) the polyhedral point of view; and (2.4) the P_4 property.

2.1. The Perfect Graph Theorem

The weak conjecture of Berge states that the following two conditions (2.1.1) and (2.1.2) are equivalent.

(2.1.1) $\alpha(H) = \gamma(H)$ (α–perfectness) for each induced subgraph H of G

(2.1.2) $\omega(H) = \theta(H)$ (γ–perfectness) for each induced subgraph H of G

This conjecture was settled by Lovász in 1972 and became known as the Perfect Graph Theorem. The key to the proof is the introduction of the following self–complementary condition (2.1.3) and the Duplication Lemma:

(2.1.3) $\alpha(H)\cdot\omega(H) \geq |H|$ for each induced subgraph H of G

Denote the set of vertices adjacent to x in G by N(x) (the set of neighbors of x). Let u be any vertex of G. Denote the graph obtained from G by adding a new vertex u' which is connected to all the neighbors of u by $G \oplus u$. $G \oplus u$ is said to be obtained from G by a ***duplication*** of u. A graph G' is said to be obtained from G by ***multiplication*** of vertices it can be obtained from G through a series of vertex duplication operations.

Lemma 2.1.4 (Duplication Lemma). If G is perfect, then $G \oplus u$ is perfect for any vertex u of G.

Theorem 2.1.5. (The Perfect Graph Theorem, Lovász [1972a]). Let G be an undirected graph. Then conditions (2.1.1), (2.1.2) and (2.1.3) are all equivalent.

2.2 Minimal Imperfect Graphs

An imperfect graph G is called ***minimal imperfect*** if it has no imperfect proper induced subgraph. Obvious properties of imperfect graphs include the following: G is connected; $\alpha(G) \geq 2$ and $\omega(G) \geq 2$. Below, we list other conditions that every minimal imperfect graph G satisfies.

1. Lovász condition (Lovász [1974b])

   $n = \alpha \cdot \omega + 1$

2. Padberg conditions (Padberg [1973,1974])

   Every vertex is in exactly ω maximum cliques

   Every vertex is in exactly α maximum stable sets

   G has exactly n maximum cliques

   G has exactly n maximum stable sets

Each maximum clique intersects all but one maximum stable set and vice versa.

Although these two conditions seem to be quite strong, they are not enough to characterize minimal imperfect graphs. To illustrate this, let us call an undirected graph G on n vertices **(α,ω)–partitionable** if $n = \alpha\cdot\omega + 1$ and for all vertices x of G $\alpha(G) = \theta(G\backslash\{x\})$, $\omega(G) = \gamma(G\backslash\{x\})$. It can be shown that G is partitionable if and only if both conditions 1 and 2 hold. Furthermore, the class of minimal imperfect graphs is properly contained in the class of partitionable graphs (Padberg [1974]), and the latter is properly contained in the class of imperfect graphs (Bland, Huang and Trotter [1979]). Hence, we have

(2.2.1) A graph G is imperfect if and only if some induced subgraph of G is partitionable.

This property immediately implies that the class of imperfect graphs belongs to NP, which is equivalent to that the class of perfect graphs belongs to coNP. Figure 1 shows a graph which is partitionable but not minimal imperfect (discovered independently by Huang [1976] and by Chvátal, Graham, Perold and Whitesides [1979]).

Figure 1. A graph G which is partitionable but not minimal imperfect

Chvátal gave a large class of partitionable graphs which is not minimal imperfect. Denote by C_n^k the undirected graph with vertices $v_1, v_2, \ldots, v_n$ such that v_i and v_j are adjacent if and only if i and j differ by at most k (all subscript arithmetic is taken modulo n). It is easy to see that the graph $C^{\omega-1}_{\alpha\omega+1}$ is an (α,ω)–partitionable graph. When $\omega = 2$, then $C^{\omega-1}_{\alpha\omega+1}$ is simply the odd hole $C_{2\alpha+1}$; when $\alpha = 2$, then $C^{\omega-1}_{\alpha\omega+1}$ is the odd antihole $\overline{C}_{2\omega+1}$. We have

Lemma. 2.2.2 (Chvátal [1976]). For any integer $\alpha \geq 3$ and $\omega \geq 3$, the partitionable graph $C^{\omega-1}_{\alpha\omega+1}$ is not minimal imperfect.

Many other properties for minimal imperfect graphs were found besides Lovász and Padberg conditions. Most of them provide forbidden subgraphs characterizations.

3. G does not have an even pair (Meyniel [1984])

Two vertices x and y of a graph are said to form an *even pair* if there is no induced odd path connecting x and y. This property implies

(2.2.3) G does not have twins (Lovász [1972a]).

where, two vertices x and y are said to be *twins* if every vertex distinct from x and y is adjacent to either both of them or to neither of them. If G contains twins, then it is substitution decomposable.

Reed (private communication) showed that property 3 holds for partitionable graphs.

4. G does not have antitwins (Olariu [1986])

Two vertices x and y are said to be *antitwins* if every vertex distinct from x and y is adjacent to precisely one of them. Note that, this property is not shared by all partitionable graphs as one can check that the top and bottom two vertices of the graph shown in Figure 1 form antitwins.

5. (Star–cutset Lemma, Chvátal [1985]). G does not have a star–cutset.

A *cutset* C in a connected graph G is a subset of vertices whose deletion leaves G disconnected. A *star–cutset* is a cutset C in which there is a vertex adjacent to every other vertex in C. By restricting star–cutsets to those consisting of some vertex along with all of its neighbors, the star–cutset lemma is reduced to the following result of Tucker [1977].

(2.2.4) G does not contain a vertex x such that $G \backslash N(x)$ is disconnected.

By the Perfect Graph Theorem, (2.2.4) is equivalent to the following result of Olaru

(2.2.5) G does not contain a vertex w such that the set of all the vertices adjacent to w induces a disconnected subgraph of the complement of G.

Given any class $\mathscr{C}$ of perfect graphs, one can enlarge this class by considering its star–closure $\mathscr{C}^*$, defined recursively as follows.

(1) If $G \in \mathscr{C}$, then $G \in \mathscr{C}^*$.

(2) If G or $\overline{G}$ has a star–cutset, and if $G\setminus\{v\} \in \mathscr{C}^*$ for all vertices v of G, then $G \in \mathscr{C}^*$.

6. The skew–partition conjecture (Chvátal [1985]).

Define a ***skew partition*** of a graph G to be a partition of V into V_1 and V_2 such that both V_1 and $\overline{V}_2$ are disconnected. Note that G has a skew partition if and only if $\overline{G}$ has. Chvátal proposed the following

The Skew–Partition Conjecture No minimal imperfect graph has a skew partition.

For graphs with at least five vertices and at lest one edge, having a star–cutset implies having a skew partition. Hence this conjecture generalizes the star–cutset lemma. Furthermore, it can be deduced from the SPGC, because an odd hole does not have a skew–partition.

2.3 Polyhedral Point of View

Padberg [1973,1974] derived many properties of minimal imperfect graphs using techniques drawn from polyhedral combinatorics.

Let A be any m x n matrix of zeros and ones having no zero column, and define the polytopes P and P_I as follows:

$$P = \{ x \in \mathbb{R}^n \mid Ax \leq e, x_j \geq 0, j = 1,\ldots,n \}$$

$$P_I = \text{conv}\{ x \in \mathbb{R}^n \mid Ax \leq e, x_j = 0 \text{ or } 1, j = 1,\ldots,n \}$$

where $e^T = (1,...,1)$ has m components all equal to one. The matrix A is called *perfect* if $P = P_I$, i.e., if the polytope P has only integral vertices. Denote by G ***the graph associated with the matrix*** A, i.e., the vertices of G correspond to the columns of A and two vertices are adjacent if their corresponding columns have at least one +1 entry in common. A matrix A is said to be a ***clique matrix*** if it contains the incidence vectors of all maximal cliques of the associated graph G. Chvátal [1975] showed that A is perfect if and only if its associated graph G is perfect.

Let A be a clique matrix of size m x n and let G be the associated graph. Denote by B a clique matrix of $\overline{G}$. Similarly, define the polytopes Q and Q_I for $\overline{G}$ as follows:

$$Q = \{ x \in \mathbb{R}^n \mid Bx \leq e, x_j \geq 0, j = 1,...,n \}$$

$$Q_I = \text{conv}\{ x \in \mathbb{R}^n \mid Bx \leq e, x_j = 0 \text{ or } 1, j = 1,...,n \}$$

where $e^T = (1,...,1)$ has components all equal to one and is dimensioned compatibly with B. Denote $\omega(G)$ by ω. Padberg showed that

Lemma 2.3.1. Let A be a clique matrix of a minimal imperfect graph G. Then $\sum_{j=1}^{n} x_j \leq \omega$ provides a facet of Q_I and $x = (1/\omega)e$ is a fractional vertex of P.

This lemma implies that every minimal imperfect graph G contains at least $|V(G)|$ maximum cliques of cardinality ω.

Lemma 2.3.2. Let A be an m n clique matrix of a minimal imperfect graph G. Then A contains an n n nonsingular submatrix A_1 whose column and row sums are all equal to ω. Furthermore, any row of A which is not in A_1 is either componentwise identical to some row of A, or has a row sum strictly less than ω.

This lemma implies that every minimal imperfect graph G has exactly $|V(G)|$

maximum cliques of size $\omega(G)$ and $|V(G)|$ maximal independent sets of size $\alpha(G)$.

Let A be a zero–one matrix of zeros and ones and G its associated graph. A is said to have ***property*** $\pi_{\beta,n}$ if the following conditions hold:

(i) A contains an n n nonsingular submatrix A_1 whose row and column sums are all equal to β.

(ii) Each row of A which is not a row of A_1 either is componentwise equal to some row of A_1 or has a row sum strictly less than β.

Padberg derived the following forbidden submatrix characterizations for perfect matrices.

Theorem 2.3.3. Let A be any zero–one matrix of size $m \times n$. Then A is perfect if and only if for $\beta \geq 2$ and $3 \leq k \leq n$, A does not contain any $m \times k$ submatrix A' having property $\omega_{\beta,k}$.

2.4 The P_4–structure

Chvátal [1984] defined a graph G to ***have the P_4–structure*** of a graph H (or G is P_4–isomorphic to H), if there is a bijection f between $V(G)$ and $V(H)$ such that a set S of four vertices in G induces a P_4 if and only if $f(S)$ induces a P_4 in H. Based on this notion, he proposed the following

Semi–Strong Perfect Graph Conjecture (SSPGC): If G has the P_4–structure of a perfect graph, then G is perfect.

The SPGC implies the SSPGC and the latter in turn implies the Perfect Graph Theorem. This semi–conjecture was shortly settled by Reed [1987] and became the semi–strong perfect graph theorem. His proof makes use of the Perfect Graph

Theorem. Below, we briefly describe the main idea of his proof. Define a ***endomorphism*** of a graph G to be a mapping f which maps V(G) into itself in such a way that f(u) and f(v) are adjacent if and only if u and v are. Suppose the SSPGC fails, then there must exist P_4–isomorphic graphs G and H such that G is perfect and H is minimal imperfect. Obviously, G is not H and, by the Perfect Graph Theorem, G is not $\overline{H}$. A contradiction can then be obtained by showing that H must have one of the following three properties which a minimal imperfect graph cannot have.

Theorem 2.4.1. Let G and H be P_4–isomorphic graphs such that G is neither H nor $\overline{H}$. Then at least one of the following holds:

(a) H contains a proper induced subgraph isomorphic to C_5.

(b) H or $\overline{H}$ has a star–cutset.

(c) H or $\overline{H}$ has a proper endomorphism.

The semi–strong perfect graph theorem suggests that it suffices to investigate the formation of P_4 in a graph to check perfection. Based on this, Chvátal [1987] invented the partner decomposition to be discussed in Section 4.

3. Current List of Perfect Graphs

Many interesting classes of perfect graphs were reported in the literature. characterizations for them range from forbidden subgraphs, representations as intersection graphs, and edge orientations. Some of these classes are perfect by definition, e.g. triangulated graphs, comparability graphs. Others are obtained from the intersection of the class of perfect graphs with certain special classes of graphs, e.g. planar perfect graphs, claw–free perfect graphs. Most of them have "good characterizations", namely, characterizations which render polynomial time recognition algorithms. Researchers are often interested in finding well characterized classes of perfect graphs which contain a number of known special classes of perfect graphs. Another line of research involves determining the most primitive classes of perfect graphs which can be composed nicely to form various larger classes of perfect graphs. This aspect will be the main topic in Section 4. We shall divide our discussion into the following two types of graphs: (3.1) special classes of perfect graphs; (3.2) graphs which satisfy the SPGC. For each class listed, we state a few equivalent characterizations.

3.1. Special Classes of Perfect Graphs

The birth of perfect graphs was closely related to the discovery of the perfectness of comparability graphs (Berge [1960]), triangulated graphs (Berge [1960], Hajös and Suranyi [1958]) and interval graphs (Berge [1960]). Since then, many other classes were discovered. A diagram depicting the relationships among some of these classes is shown in Figure 3. We note here that, although it seems plausible to investigate only the larger classes of perfect graphs such as weakly triangulated graphs or strongly perfect graphs, the alternate representations of the smaller classes are also worth discussing. Figure 2 depicts the relationships among some of these classes. Figure 3 gives some sample graphs illustrating the differences among some classes.

Figure 2. The Hasse diagram of the containment relationships among some classes of perfect graphs

Figure 3. Examples showing the differences among these special classes

In the following, consider a finite, undirected graph G = (V,E) without loops or multiple edges:

1. **Interval graph**: The intersection graph of a family of intervals.

   This class was first noted by Hajos [1957].

   An equivalent characterization (Gilmore and Hoffman [1979]): G is triangulated and $\overline{G}$ is a comparability graph.

   Forbidden subgraphs can be found in Lekkerkerker and Boland [1962] (also see Duchet [1984]).

2. **i–triangulated graph** (Gallai [1962]): Each odd cycle of length > 3 has a set of chords which form with the cycle a planar graph whose unbounded face is the exterior of the cycle and whose bounded faces are all triangles.

   Equivalent Characterizations:

   (a) Every odd cycle of length greater than 3 has at least two non–crossing chords.

   (b) Every odd cycle of length k has k–3 chords that do not cross one another.

3. **Comparability graph**: there exists a transitive acyclic orientation ($a \to b$ & $b \to c$ implies that $a \to c$) on the edges of G.

   An equivalent characterization (Rotem and Urrutia [1983]): The complement of G is the intersection graph of the graphs of n continuous functions F_i: $(0,1) \to \mathbb{R}$.

Forbidden induced subgraph characterizations can be found in Gallai [1967] (or see Duchet [1984]).

Permutation graphs, a special class of comparability graphs, can be characterized as: G and $\overline{G}$ are both comparability graphs (Dushnik and Miller [1941]). A special class of permutation graphs, the class of P_4–free graph (also called **cograph**) satisfies that: for any induced subgraph H of G, either H or $\overline{H}$ is not connected. Corneil, Perl and Stewart [1985] gave a linear time algorithm for constructing a substitution decomposition tree of a cograph.

4. **Parity graph** (Olaru and Sachs [1970]): Every odd cycle of length greater than 3 contains at least two crossing chords.

An equivalent characterization: for every pair of vertices x and y in G. all induced paths connecting x and y have the same (odd, even) parity.

This class of graphs includes the class of bipartite graphs and P_4–free graphs.

5. **Tolerance graph** (Golumbic, Monma and Trotter [1984]): there exist a family $F = \{I_x \mid x \in V\}$ of closed intervals and a set $T = \{t_x \mid x \in V\}$ of positive numbers satisfying that $(x,y) \in E$ iff the length of $I_x \cap I_y \geq \min \{t_x,t_y\}$

A tolerance graph is weakly triangulated and its complement is perfectly orderable.

6. **Triangulated graph** (also called ***chordal*** graph): there exist no induced even cycles of length ≥ 4.

A ***simplicial*** vertex u in G is one whose neighbors form a clique. A ***cutset*** C in a connected graph G is a subset of vertices whose deletion disconnects G. A cutset is said to be ***minimal*** if it contains no proper cutset. An ***R–orientation*** is an acyclic orientation satisfying that if $a \rightarrow c$ and $b \rightarrow c$, then $(a,b) \in E(G)$.

Equivalent characterizations:

(a) G is the intersection graph of subtrees of a tree whose vertices correspond to the maximal cliques of G. (Buneman [1974], Gavril [1979])

(b) Every minimal cutset of G is a clique. (Dirac [1961])

(c) Every induced subgraph of G has a simplicial vertex. (Dirac [1961], Lekkerkerker and Boland [1962])

(d) G admits an R–orientation. (Rose [1970])

7. **Meyniel graph** (Meyniel [1976]): Every odd cycle of length greater than 3 contains at least two chords.

This class of graphs include the class of i–triangulated graphs (Gallai [1962]).

8. **Perfectly orderable graph** (Chvátal [1984]): There exists a ***perfect linear order*** $v_1, \ldots, v_n$ on the vertices so that a minimum coloring can be obtained by assigning to each v_i the smallest positive integer that is assigned to none of its neighbors with index less than i.

Consider a linear order $<$ on the vertices of G. A P_4 abcd in G with edges (a,b), (b,c), (c,d) is said to form a ***obstruction*** if $a < b$, $d < c$. A linear order is said to be ***admissible*** if it creates no obstruction.

An equivalent characterization: G has an admissible order.

This class of graphs includes the class of comparability graphs, triangulated graphs and cotriangulated graphs.

9. **Weakly triangulated graph** (Hayward [1985]): G contains no C_n or $\overline{C}_n$ for all $n \geq 5$.

Hayward showed that if G is a weakly triangulated graph with at least three vertices, then G or $\overline{G}$ has a star–cutset. Weakly triangulated graphs are preserved under substitution. A vertex x is said to be ***dominated*** by a vertex y if every vertex z (different from x and y) that is adjacent to x is also adjacent to y. A ***homogeneous*** *set* M in G is a proper subset of V(G) such that H has at least two vertices, and every vertex in V(G)\M is adjacent to either all or none of the vertices of M. Hayward gave an interesting weakly triangulated graph W, which is self complementary, without clique cutset nor a homogeneous set, without dominated vertices and is not strongly perfect.

10. **Alternately orientable graph** (Hoang [1987]): there exists an orientation such that the directions alternate on every induced cycle of length at least four.

If G is an alternation graph, then G is a comparability graph or else it has a star–cutset.

11. **Strongly perfect graph** (Berge and Duchet [1984]): Each induced subgraph H contains a stable set which meets all maximal cliques in H.

Another characterization: no two families $\mathscr{C} = \{C_1,...,C_k\}$ and $\mathscr{D} = \{D_1,...,D_r\}$ of maximal cliques (with possible repeated cliques) satisfy $|\mathscr{C}| = |\mathscr{D}|$ and $|\mathscr{C}_x| > |\mathscr{D}_x|$ for all $x \in V$.

This class includes the classes of P_4–free graphs, comparability graphs, triangulated graphs, Meyniel graphs (Ravindra [1984]), cotriangulated graphs, and perfectly orderable graphs.

12. **Quasi–Parity graph** (Meyniel [1987]): if H is an induced subgraph of G which is neither a clique nor a stable set, then either H or its complement contains an even pair (no induced odd path connecting x and y).

G is said to be **strict quasi–parity** if every induced subgraph of G contains an even pair or else is a clique. These graphs are perfect because no minimal imperfect graph can contain an even pair.

P. Hell (private communication) remarked that, given any class $\mathscr{C}$ of perfect graphs, one can enlarge it to a class $\mathscr{C}^{QP}$ defined as follows. A graph $G \in \mathscr{C}^{QP}$ if, for every induced subgraph G' of G, one of the following holds:

(i) G' belongs to $\mathscr{C}$.

(ii) There exists in G' or in $\overline{G}'$ two vertices x, y which are not linked by an odd induced path.

13. **Murky graph** (Hayward [1988]): G contains neither C_5, P_6, nor $\overline{P}_6$.

We now follow the notation of Hayward. A graph is called ***unbreakable*** if it has more than two vertices and if neither the graph nor its complement has a star cutset. Define a ***mirror partition*** [R,S] of a graph G to be a partition of the vertices into sets $R = \{r_1,...,r_t\}$ and $S = \{s_1,...,s_t\}$ such that

(1) G[R] and G[S] are P_4–free, and

(2) $(r_i,r_j) \in E \Leftrightarrow (s_i,s_j) \in E \Leftrightarrow r_i$ misses $s_j \Leftrightarrow s_i$ misses r_j, for $1 \leq i < j \leq t$.

A graph that has a mirror partition is called a ***mirror graph***. Let L_8 and L_9 denote the line graphs of $K_{3,3}\backslash e$ and $K_{3,3}$, respectively. Hayward showed that if G is an unbreakable Murky graph, then G is L_8, L_9 or a mirror graph.

14. **Opposition graph** (Olariu [1987]): there exists an acyclic orientation of G so that no P_4, wxyz, with edges (w,x), (x,y) and (y,z) has both $w \rightarrow x$ and $y \rightarrow z$.

If G is an opposition graph, then G is either bipartite or else its complement has a star–cutset.

15. ***PI* graphs** (Corneil and Kamula [1987]): Each vertex is represented by a point on a line and an interval on another line (which form a triangle); two vertices are adjacent iff their corresponding triangles intersect.

This class includes both the permutation graphs and the interval graphs. An extension of *PI* graphs is the class of *II* graphs (discovered independently by Dagan, Golumbic and Pinter [1986], which they called **trapezoid graphs**): each vertex is represented by an interval on each line (which form a trapezoid) and two vertices are adjacent iff their corresponding trapezoids intersect. The class of *II* graphs is contained in the class of weakly triangulated graphs.

3.2 Graphs Which Satisfy the SPGC

The SPGC has been verified for many special classes of graphs. The perfect subclasses of these special graphs give rise to new classes of perfect graphs. We list several important subclasses below. A graph is called Berge if it contains no odd holes or odd antiholes. The following notations are needed. A ***claw*** is a $K_{1,3}$, namely, a set of four vertices {w,x,y,z} with edges (w,x), (w,y) and (w,z). Such a claw is said to be centered at w. A ***diamond*** is a K_4 with one edge deleted. A bull and a dart are depicted in Figure 5.

Figure 4. Three graphs: diamond, bull and dart

1. **Planar perfect graph** (Tucker [1972]): A planar Berge graph.

    Planar graphs is the first special class containing imperfect graphs for which the SPGC is verified. The proof by Tucker [1972] yields a coloring algorithm. Recently, Hsu [1987] gave a recognition algorithm which simultaneously solved all four optimization problems (Hsu [1988]).

2. **Claw–free perfect graph** (Parthasarathy and Ravindra [1976]): A Berge graph without claws.

    The class of claw–free graphs became well–known after Minty's maximum independent set algorithm (Minty [1980]). The first proof by Parthasarathy and Ravindra [1976] considered minimal imperfect claw–free graphs. Note that the problems of finding $\omega(G)$, $\gamma(G)$ and $\theta(G)$ are all NP–hard for general claw–free graphs. Hsu and Nemhauser [1984] solved these latter problems polynomially on their perfect subclass based on the bipartite matching algorithm.

3. **Perfect 3–Chromatic graph** (Tucker [1977]): A Berge graph without K_4.

The first proof by Tucker [1977] considered minimal imperfect 3–chromatic graphs. Recently, he gave anther proof based on a coloring algorithm (Tucker [1987]).

4. **Perfect toroidal graph** (Grinstead [1978]): A Berge toroidal graph.

A ***toroidal graph*** is a graph which can be drawn on a torus so that no two edges intersect. The key in verifying the SPGC is the following: if G is minimal imperfect toroidal, then either $\omega(G) < 4$ or G is regular of degree six and triangulated the torus.

5. **Diamond–free** (($K_4 \setminus e$)–free) perfect graph (Parthasarathy and Ravindra [1979], Tucker [1984]): A Berge graph without diamonds.

Original proofs considered minimal imperfect diamond–free graphs. Tucker [1987] recently gave a coloring algorithm. Fonlupt and Zemirline [1987] gave a recognition algorithm.

6. **Bull–free perfect graph** (Chvátal and Sbihi [1987]): A Berge graph without bulls.

7. **Dart–free perfect graph** (Sun [1988]): A Berge graph without darts.

8. **Gallai perfect graph** (Sun [1988]): Gal(G) contains no odd holes.

Given any graph G, define a graph Gal(G) by letting the vertices of Gal(G) be the edges of G, and making two vertices of Gal(G) adjacent if and only if the corresponding two edges of G share an endpoint and their other two endpoints are nonadjacent in G (namely, these two edges form a P_3).

The proof uses the following important property: If a Gallai–perfect graph G contains an induced dart, then G contains a star–cutset or an even pair.

Techniques used in verifying the SPGC for special classes have been centered around the neighborhood structure of a minimal imperfect graph in those classes. Hsu [1984] summarized the proofs for 3–chromatic, claw–free and diamond–free graphs and concluded that the SPGC is true for graphs in which each vertex is one of the following five types: Define a vertex u to be a

(1) b–vertex if N(u) is bipartite

(2) $\overline{b}$–vertex if the complement of N(u) is bipartite

(3) m–vertex if N(u) is complete multipartite

(4) $\overline{m}$–vertex if the complement of N(u) is multipartite

(5) $\overline{m}'$–vertex if each of its neighbor is an $\overline{m}$–vertex.

In a 3–chromatic graph every vertex is a b–vertex. In a claw–free graph every vertex is a $\overline{b}$–vertex. In a diamond–free graph every vertex is an $\overline{m}$–vertex. This result was later used by Sun [1988] to prove that dart–free graphs satisfy the SPGC.

Chvátal [1976] proved that if a minimal imperfect graph G contains a spanning subgraph isomorphic to C_n^k for some $k \geq 1$, then G is an odd hole or an odd antihole. Giles, Trotter and Tucker [1984] strengthened this result to show that if, for each u in a minimal imperfect graph G, the partition of V–{u} into $\alpha(G)$ stable sets has at least two members containing a single neighbor of u, then G is an odd hole or an odd antihole. This result also implies the validity of the SPGC for claw–free graphs.Finally, define a class to be ***complete for the SPGC*** iff the truth of the SPGC on this restricted class implies that the SPGC is true in general. By applying perfection–preserving composition operations, Corneil [1986] showed that the following classes are complete for the SPGC: k–connected graphs for any positive integer k, graphs which are both eulerian and hamiltoonian, self–complementary graphs and regular graphs.

4. Algorithmic Aspects of Perfect Graphs

The definition of perfect graphs involves the four most important parameters in graph optimization problems (hereafter, referred to as the **four optimization problems**). The strong duality relationships on $\alpha(G)$, $\theta(G)$, and $\omega(G)$, $\gamma(G)$ provide invaluable information for efficiently computing these parameters. In addition, perfect graphs are hereditary and therefore, suitable for divide–and–conquer type of algorithms. These properties can probably explain the recent explosion of algorithmic interests and results on perfect graphs.

At the birth of perfect graphs in 1960's, several special classes of perfect graphs were found and their four optimization problems solved. However, these classes of graphs are, by definition, free of odd holes or odd antiholes and the SPGC is trivially true for them. The first nontrivial proof of the SPGC was presented by Tucker [1972] on the class of planar graphs. His proof provides an algorithm for coloring planar perfect graphs. Later, Pathasarathy and Ravindra [1976] showed that the SPGC holds for the class of claw–free graphs. Minty [1980] gave a polynomial algorithm for the maximum independent set problem on this class. Hsu [1979] solved the other three optimization problems on its perfect subclass (these three problems are NP–hard on general claw–free graphs). In 1980, Grötschel, Lovász and Schrijver [1980] finally discovered polynomial algorithms for the four optimization problems on general perfect graphs. Their algorithms are based on a separation concept of the ellipsoid algorithm and bears little relationships with the graphical structure of perfect graphs. Although their algorithms are of theoretical interest, their results pave the way for future research on more efficient combinatorial algorithms for perfect graphs.

Another important problem, the recognition of general perfect graphs, remains open. The situation on special classes of perfect graphs is much better. Polynomial (in many cases, linear) time algorithms have been designed for recognizing triangulated graphs (Rose, Tarjan and Lueker [1976]), interval graphs (Booth and Lueker [1976]), comparability graphs (Golumbic [1977], Spinrad [1985]), permutation graphs (Even, Pnueli and Lempel [1972]), cographs (Corneil, Perl and Stewart [1985]), parity graphs (Burlet and Uhry [1984]), Meyniel graphs (Burlet and Fonlupt [1984]) and etc. A common nature of these algorithms is that they are all based on representations which yield either graph decompositions or special vertex orderings. Again, for these special classes of graphs the SPGC holds trivially, because of the absence of odd holes and antiholes by definition. In 1987, Hsu provided an $O(n^3)$ algorithm for recognizing planar perfect graphs, which is equivalent to identifying odd holes in planar graphs. This is the first algorithm for recognizing odd holes in a nontrivial (meaning that there are graphs containing odd holes) class of graphs. Similar recognition algorithms have also been obtained by Chvátal and Sbihi [1987] on claw–free perfect graphs, by Fonlupt and Zemirline [1987] on $(k_4\backslash e)$–free perfect graphs. Hsu [1987a] gave a summary on the existing decomposition operations for perfect graphs.

We shall first discuss the general optimization algorithms of Grötschel, Lovász and Schrijver in Section 4.1. We then discuss some composition and decomposition operations on perfect graphs in Section 4.2. These operations are shared by many combinatorial recognition and optimization algorithms for perfect graphs. Section 4.3 is devoted to combinatorial algorithms. Finally, we cover some related problems and applications.

4.1. The Ellipsoid algorithms of Grötschel, Lovász and Schrijver

The algorithms of Grötschel et al. use the ellipsoid method. We assume the reader has a basic knowledge of this method (see Gács and Lovász [1981]). Define a ***convex body*** K to be a closed, bounded fully dimensional, and convex subset of $\mathbb{R}^n$, $n \geq 2$; specifically, there exist two rational numbers $0 < r \leq R$, and a vector $a_0 \in K$

such that $S(a_0,r) \subseteq K \subseteq S(a_0,R)$, where $S(a_0,s) = \{ x \in \mathbb{R}^n \mid \|x - a_0\| \leq s \}$ denotes the ball with center a_0 and radius s. Denote the convex body by the quintuple $(K;n,a_0,r,R)$.

The following two related problems are of particular interest.

Optimization Problem. Given a vector $c \in \mathbb{Q}^n$ and a rational number $\epsilon > 0$, find a vector $y \in \mathbb{Q}^n$ such that $d(y,K) \leq \epsilon$ and $c^T x \leq c^T y + \epsilon$ for all $x \in K$ (i.e. y is almost in K and almost maximizes $c^T x$ on K).

Separation Problem. Given a vector $y \in \mathbb{Q}^n$ and a rational number $\delta > 0$, conclude with one of the following:

(1) asserting that $d(y,K) \leq \delta$ (i.e. y is almost in K)

(2) finding a vector $c \in \mathbb{Q}^n$ such that $\|c\| \geq 1$ and for every $x \in K$, $c^T x \leq c^T y + \delta$ (i.e. finding an almost separating hyperplane).

Let $\mathbb{R}^n_+$ be the nonnegative orthant and $K \subseteq \mathbb{R}^n$ be a convex body such that there are reals r and R, $0 < r < R$, with

(a) $\mathbb{R}^n_+ \cap S(0,r) \subseteq K \subseteq \mathbb{R}^n_+ \cap S(0,R)$

(b) $0 \leq x \leq y \in K \Rightarrow x \in K$

The anti–blocker A(K) of K is defined by

$$A(K) := \{ y \in \mathbb{R}^n_+ \mid y^T x \leq 1 \text{ for every } x \in K \}$$

If $\mathscr{K}$ is a class of convex bodies satisfying (a) and (b) we set $A(\mathscr{K}) = \{ A(K) \mid K \in \mathscr{K} \}$.

Theorem 4.1.1. Let $\mathscr{K}$ be a class of convex bodies satisfying (a) and (b). Then the optimization problem for $\mathscr{K}$ can be solved in polynomial time if and only if the optimization problem for $A(\mathscr{K})$ can be solved in polynomial time.

We shall only briefly describe their approach for solving the unweighted stable set problem for perfect graphs. They consider the following two classes of convex bodies.

For every subset W of vertices in G, denote by x^W the incidence vector of W, i.e. $x_v^W = 1$ if $v \in W$ and $x_v^W = 0$ if $v \notin W$. Then

$$P(G) := \text{conv}\{ x^W \in \mathbb{R}^n \mid W \subseteq V(G) \text{ is a stable set of } G \}$$

is called the ***stable set polytope*** of G.

$$P^*(G) := \{ x \in \mathbb{R}^n \mid x_v > 0 \text{ for all } v \text{ in } V(G) \text{ and } \sum_{v \in C} x_v \leq 1 \text{ for all cliques } C \subseteq V(G) \}$$

is called the ***fractional stable set polytope*** of G, which contains P(G). Define

$$\alpha^*(G) := \max\{ \sum_{v \in V} x_v \mid x \in P^*(G) \}$$

to be the ***fractional stability number***.

Clearly, every weighted stable set problem on G can be solved as a linear programming problem over P(G). The LP–solution over $P^*(G)$ provides an upper bound $\alpha^*(G)$ for the weight of the optimal stable set in G. Fulkerson [1973] has shown the following (see also Chvátal [1975]).

Theorem 4.1.2. $P(G) = P^*(G)$ if and only if G is perfect.

Hence for perfect graphs, $\alpha(G) = \alpha^*(G)$. However, it is difficult to find a polynomial separation algorithm for P(G). Instead, they investigate the Shannon capacity of perfect graphs.

Denote by $G \times H$ the ***cartesian product*** of the graph G and H, i.e. $V(G \times H) = V(G) \times V(H)$ and two vertices (u,v), (u',v') in $V(G \times H)$ are adjacent if and only if u is adjacent to u' and v is adjacent to v'. Let G^k denote the cartesian product of k copies of G. The Shannon capacity of a graph G is defined to be

$$\mathscr{O}(G) := \sup_k \sqrt[k]{\alpha(G^k)} \quad \text{(Shannon [1956])}$$

It is easy to see that $\alpha(G) \leq \mathcal{S}(G)$. Shannon showed that $\mathcal{S}(G) \leq \alpha^*(G)$. Hence, for perfect graphs, $\alpha(G) = \mathcal{S}(G) = \alpha^*(G)$. But, $\mathcal{S}(G)$ is difficult to compute in general. Lovász introduced a parameter, called $\mathcal{N}(G)$, which is an upper bound for $\mathcal{S}(G)$.

Let G be a graph and assume that its vertices are labeled 1, 2, ..., n. A system $(u_1, \ldots, u_n)$ of vectors in a real vector space is an ***orthonormal representation*** of G if the u_i's are orthogonal and of length 1. Let $\mathcal{U}(G)$ be the set of all orthonormal representations of G, and U be the set of vectors of unit length, then set

$$\mathcal{N}(G) := \min_{(u_1,\ldots,u_n)\in \mathcal{U}(G)} \; \min_{c\in U} \; \max_{1\leq i\leq n} \frac{1}{(c^T u_i)^2}$$

This quantity can also be characterized as a maximum of the sum of the entries of certain matrices representing G. Denote the trace of a matrix B by tr(B). Define

$$\mathcal{B}(G) := \{ B = (b_{ij}) \mid B \text{ is a symmetric positive semidefinite (n,n)–matrix with } tr(B) = 1 \text{ such that } b_{ij} = 0 \text{ iff } (i,j) \in E(G) \}$$

Lovász showed that

$$\mathcal{N}(G) = \max \{ \sum_{i,j=1}^{n} b_{ij} \mid B \in \mathcal{B}(G) \}.$$

For perfect graphs, we have $\alpha(G) = \mathcal{S}(G) = \mathcal{N}(G) = \alpha^*(G)$. Grötschel et al. provide a polynomial separation algorithm for the class of positive semidefinite matrices $\mathcal{B}(G)$, which enables them to calculate $\mathcal{N}(G)$ and hence, $\alpha(G)$.

4.2 Decomposition Operations

One of the approaches in analyzing perfect graphs is to investigate perfection–preserving operations which either decompose perfect graphs into highly structured components or generate large perfect graphs from the composition of smaller ones. The motivation for decomposing perfect graphs is to identify those primitive, highly structured classes of perfect graphs (e.g. comparability graphs, line graphs of bipartite graphs) from which general perfect graphs can be constructed. Hsu [1987b] conjectured that every perfect graph can be composed from components that are either comparability graphs or line graphs of bipartite graphs.

One of the important operations, implied by the Perfect Graph Theorem, is "complementation". In applying the decomposition operation described in this section, we not only check a given graph, but also check its complement. A number of operations satisfy the interesting property of being ***self-complementary***, namely, the graphical structure associated with the operation exists in graph G if and only if it exists in $\overline{G}$.

Most of the operations discussed in this section preserve perfectness for both decomposition and composition, which are very useful for recognition algorithms. For each operation, there is an associated graphical structure, whose existence in a graph G enables us to apply the operation on G. Depending on the nature of these operations, we can classify them based on the following considerations: (1) formation of induced cycles; (2) P_4-structure; (3) graph reduction. At then end of this section, we pose some open problems.

4.2.1 Formation of Induced Cycles

The idea can be drawn from regular graph connectivity. It is well-known that the biconnectivity definition of graphs is related to " cycles" in that two edges in different biconnected components are not contained in any simple cycle. For perfect graphs, it appears that the corresponding notions are induced cycles (holes) and their complements (antiholes). Since odd holes and odd antiholes are imperfect, none of the perfection-preserving operations can generate odd holes or odd antiholes. In fact, most of these operations do not create any holes at all. We shall discuss several classical operations of this kind and summarize them at the end.

Define a *cutset* in a connected graph G to be a set of vertices in G whose deletion disconnects G. The operations described in this section involve certain vertex partitioning and cutsets.

4.2.1.1 Operations Which Do Not Create Holes

We describe several perfection–preserving composition operations which do not generate holes or antiholes. The reader will find that certain operations are special cases of others. We include them because there exist different generalizations and applications for these operations.

1. Clique identification

Given a graph G_1 with a clique K_1, another graph G_2 with a clique K_2 such that $|K_1| = |K_2|$ and a one–to–one mapping f from K_1 to K_2, one can construct a new graph G from G_1 and G_2 by identifying each vertex v of K_1 with f(v) of K_2. G is said to be obtained from G_1 and G_2 through clique identification. On the other hand, there is a natural reverse decomposition operation. Suppose a connected graph G contains a clique cutset K. Let $H_1, \ldots, H_k$ be the connected components of $G \setminus K$. Let G_i be the subgraph of G induced on $V(H_i) \cup K$, $i = 1, \ldots, k$. Then we say G can be decomposed into $G_1, \ldots, G_k$.

It is well–known that clique identification is a perfection–preserving operation (one can check that $\omega(G) = \max(\omega(G_1), \omega(G_2))$). Decomposition based on clique cutsets is often useful in recognizing special classes of perfect graphs and in solving their optimization problems. Whitesides [1984] has developed polynomial algorithms based on such decomposition. Burlet and Fonlupt [1984] showed that i–triangulated graphs can be composed from basic i–triangulated graphs through clique identification.

2. Join

This operation is more general than substitution. Let G_1, G_2 be two disjoint graphs and v_1, v_2 be two distinct vertices in them, respectively. Define $G_1 \cdot G_2$ be the graph with

$$V(G_1 \cdot G_2) = (V(G_1) \cup V(G_2)) \setminus \{v_1, v_2\}$$

$$E(G_1 \cdot G_2) = E(G_1 \setminus \{v_1\}) \cup E(G_2 \setminus \{v_2\}) \cup \{ (x,y) \mid (x,v_1) \in E(G_1), (y,v) \in E(G_2) \}$$

Thus, $G_1 \cdot G_2$ is obtained by joining every vertex in $N(v_1)$ to every one in $N(v_2)$. Bixby [1984] showed that this operation is perfection–preserving by constructing efficient covering algorithms. This composition is one instance of a more general construction studied in Cunningham and Edmonds [1980].

Conversely, if a graph G satisfies that there exists a partition of V(G) into V_0, V_1, V_2 and V_3 with $|V_0 \cup V_1| \geq 2$, $|V_2 \cup V_3| \geq 2$ such that every vertex in V_1 is adjacent to every one in V_2, no vertex in V_0 is adjacent to any one in $V_2 \cup V_3$ and no vertex in V_3 is adjacent to any one in $V_0 \cup V_1$, then we say G ***contains a join*** and can be decomposed by the ***join decomposition*** into G_1 and G_2, where G_1 is a subgraph induced on $V_0 \cup V_1 \cup \{v_2\}$, and G_2 is a subgraph induced on $\{v_1\} \cup V_2 \cup V_3$, where v_1, v_2 are arbitrary vertices in V_1, V_2, respectively.

Cunningham [1982] gave an $O(n^3)$ algorithm for finding the join decomposition tree of a graph. Gabor, Hsu and Supowit [1985] reduced its complexity to $O(m \cdot n)$. Burlet and Uhry [1984] showed that parity graphs can be composed from bipartite graphs and cliques through join composition.

If we set V_0 to be the empty set throughout the last paragraph , then the join decomposition reduces to the ***substitution decomposition***. The substitution decomposition plays an important role in the study of comparability graphs: a comparability graph containing no substitution is uniquely transitively orientable. Spinrad [1987] has shown that the substitution decomposition tree of a graph can be obtained in $O(n^2)$ time.

3. Amalgamation

Burlet and Fonlupt [1984] developed a decomposition algorithm for recognizing Meyniel graphs. This operation is called the amalgamation, which is a generalization of the join and a special case of the clique identification. We describe the corresponding composition operation as follows.

Let G_1, G_2 be two disjoint graphs. Let K_1, K_2 be two cliques of the same size and v_1, v_2 be two distinguished vertices in G_1, G_2, respectively such that

(a) $K_1 \subsetneq N(v_1)$, $K \subsetneq N(v_2)$.

(b) Every vertex in K_i is adjacent to every other neighbor of v_i in G_i, $i = 1,2$.

(c) $N(v_1) = K_1 \Leftrightarrow N(v_2) = K_2$.

Form a new graph G, called the ***amalgam*** of G_1 and G_2, by identifying each v in K_1 with a distinct vertex in K_2, connecting every vertex in $N(v_1)\backslash K_1$ with every one in $N(v_2)\backslash K_2$ and deleting v_1, v_2. Conversely, if a graph G can be formed as the amalgam of two graphs G_1 and G_2, then we say G can be decomposed by the amalgam decomposition into G_1 and G_2. When $K_1 = K_2 = \emptyset$, the amalgamation reduces to the join. When $N(v_1)\backslash K_1 = N(v_2)\backslash K_2 = \emptyset$, the amalgamation reduces to the clique identification.

A ***basic Meyneil graph*** G is a connected graph whose V(G) can be partitioned into A, K and S such that

(a) A induces a 2–connected bipartite graph, K is a clique, S is a stable set.

(b) $x \in A$, $y \in K \Rightarrow (x,y) \in E(G)$; each $x \in S$ is adjacent to at most one vertex in A.

Burlet and Fonlupt [1984] showed that using the amalgamation, Meyniel graphs can be decomposed into basic Meyniel graphs.

In fact, one can further decompose a basic Meyniel graph G through its complement $\overline{G}$. Separating $\overline{G}$ along the clique cutset S results in two induced subgraphs G_1 and G_2 of G, where G_1 is a subgraph induced on $A \cup K$ and G_2 is a subgraph induced on $K \cup S$. Now, G_1 can be decomposed by the substitution decomposition into a bipartite graph induced on A and a clique K. G_2 can be decomposed by clique cutsets into a collection of cliques. If we regard the substitution decomposition as a special case of the amalgamation and regard a graph G to be amalgam decomposable if either G or $\overline{G}$ is the amalgam of two smaller graphs, then we have

Theorem 4.2.2. Meyniel graphs can be decomposed by the amalgamation into 2–connected bipartite graphs and cliques.

4. Generalized Join

The substitution decomposition is self–complementary. However, its generalization, the join operation, is not. Following Hsu [1987], we describe here a self–complementary operation which includes the join. A graph G is said to have a ***generalized join*** if there exists a partition of V(G) into two collection of subsets V_1^1, V_1^2, ..., V_1^t (whose union is called V_1) and V_2^1, V_2^2, ..., V_2^t (whose union is V_2) such that every vertex in V_1^j, j = 1,...,t+1, is adjacent to all vertices in $\bigcup_{m=1}^{t+1-j} V_2^m$ but no other in V_2; and every one in V_2^j, j = 1,...,t+1 is adjacent to all vertices in $\bigcup_{m=1}^{t+1-j} V_1^m$ but no other in V_1.

An equivalent condition based on forbidden configuration is that there exists a partition of V(G) into V_1 and V_2 such that

(4.2.1) there do not exist vertices x_1, y_1 in V_1, x_2, y_2 in V_2 with (x_1,x_2), $(y_1,y_2) \in E(G)$ but (x_1,y_2), $(x_2,y_1) \notin E(G)$.

From this characterization, it is easy to see that G contains a generalized join if and only if $\overline{G}$ contains one.

If G contains a generalized join as described above, then we say G can be decomposed by the generalized join decomposition into G_1 and G_2, where G_1 is a subgraph induced on $V_1 \cup \{v_2^1, v_2^2, ..., v_2^t\}$ with all edges among the v_2^t's removed and G_2 is a subgraph induced on $V_2 \cup \{v_1^1, v_1^2, ..., v_1^t\}$ with all edges among the v_1^t's removed, where v_i^j is an arbitrary vertex in V_i^j. Conversely, we say that G is composed from G_1 and G_2 by the generalized join composition.

The problem of testing whether there exists a generalized join in a graph is still open.

5. Generalized 1–separation

We describe in this section a generalization of the amalgamation by combining the clique identification and the generalized join. A connected perfect graph G is said to be *1–separable* if the following partitioning applies either to G or to its complement: V(G) can be partitioned into three subsets V_1, K, and V_2 such that K is a clique, neither V_1 nor V_2 is an independent set, (4.2.1) holds for V_1, V_2 and the following is satisfied:

(4.2.3) there does not exist an edge (x_1,x_2) connecting $x_1 \in V_1$ to $x_2 \in V_2$ such that neither (v,x_1) nor (v,x_2) belongs to E(G) for any $v \in K$.

Let G_1 (respectively, G_2) be the subgraph obtained by deleting all edges with both ends in V_2 (respectively, V_1) from G. G is said to be decomposed by the 1–separation into G_1 and G_2. Conversely, we say that G is composed from G_1 and G_2. It is easy to verify that such a composition does not create any hole or antihole and that it preserves perfection. Hence, no minimal imperfect graph can contain a partition satisfying (4.2.1) and (4.2.3).

The problem of recognizing the structure of 1–separation is open just as that of the generalized join.

4.2.1.2 Operations which create only even holes

Define a graph G to be *1–inseparable* if it cannot be decomposed by the 1–separation described above. Given a 1–inseparable perfect graph G, we consider "2–separation" operations which yields components that allow induced cycles of G to "cross" these components. A formal description of a general 2–separation is quite involved (can be found in Hsu [1987a]). We shall describe the basic idea and give a prototype example. We say that two disjoint subset of vertices U_1 and U_2 in G form a ***partial join*** if every vertex in U_1 is adjacent to every one in U_2. Consider the following analogy. Recall that 3–connected components of a 2–connected graph are obtained by repeatedly separating the graph using cutsets consisting of two vertices. Similarly, we shall apply our 2–separation to those 1–inseparable perfect graphs using separating structures consisting of two parts $\{C^1,C^2\}$, where C^1 can be one the following: a vertex, a 2–clique or the edges of a partial join and C^1 can be either a vertex or edges of a partial join. Since operations involving a partial join can be reduced to the case of a single vertex, we shall concentrate on 2–clique cutsets.

1. $C^1 = \{x\}, C^2 = \{y\}$.

Let $C = \{x,y\}$. If $(x,y) \in E(G)$, then $\{x,y\}$ is a clique cutset. Hence, assume $(x,y) \notin E(G)$. Because two induced paths connecting x, y in different components must form an even hole, every induced path connecting x, y in G must have the same parity. Let $H_1, \ldots, H_k$ be the connected components of $G \backslash C$. Let the induced subgraph on $V(H_i) \cup C$ be G_i. If x is oddly related to y, then G can be decomposed into $G_1', \ldots, G_k'$, where G_i' is the subgraph G_i with the additional vertex x_i, y_i and edges (x,x_i), (x_i,y_i) and (y_i,y). If x is evenly related to y, then G can be decomposed into $G_1'', \ldots, G_k''$, where G_i'' is the subgraph G_i with an additional vertex x_i adjacent to both x and y (as shown in Figure 5). These added vertices not only are used to preserve perfection, they also play an important role in the optimization algorithms discussed below.

Figure 5. A case for 2–separation

To illustrate how to solve the optimization problems using the decomposition scheme, we consider the maximum weight independent set (MWIS) problem. To simplify our discussion, assume G is decomposed by the cutset C into two components G_1 and G_2 and each vertex v of G is assigned a weight w(v). Furthermore, assume the MWIS on G_1 can be solved in polynomial time. We show that after suitable modification of weights (denoted by w') of vertices in G_2, a MWIS P_G of G can be obtained as the union of a MWIS P_{G_1} of G_1 and a MWIS P_{G_2} of G_2 relative to w' such that $P_{G_1} \cap C = P_{G_2} \cap C$. Thus, our MWIS algorithm on G is reduced to that on G_2 relative to w'.

Denote a MWIS of a graph G by P_G and its weight by $w(P_G)$. Define the ***marginal weight*** $w_G^C\{x,y\}$ ***of*** $\{x,y\}$ ***relative to*** C in G to be

$$w(x) + w(y) + w(P_{G\setminus(N(x) \cup N(y) \cup C)}) - w(P_{G\setminus C})$$

which is the difference between the weight of a MWIS in G containing both x and y, but no other vertex in C, and the weight of a MWIS in $G\setminus C$. It is possible for this quantity to be negative.

Two important relationships between path parity and the weights of independent sets in general graphs are stated in the following lemma.

Lemma 4.2.4. If x is evenly related to y, then $w_G^C\{x,y\} \geq w_G^C(x) + w_G^C(y)$. If x is oddly related to y, then $w_G^C\{x,y\} \leq w_G^C(x) + w_G^C(y)$.

To compensate for the deletion of G_1 when we reduce the MWIS problem to G_2, a quantity g is introduced in the following

Theorem 4.2.5. Consider the following two cases:

(a) x is connected to y in G_2 through degree–2 vertices x_1 and y_1. Without loss of generality, assume $w(x_1) \geq w(y_1)$. Let $w'(x) = w_{G_1}^C(x) + g - h$, $w'(x_1) = w(x_1) + g$, $w'(y_1) = w(y_1) + h$, $w'(y) = w_{G_1}^C(y)$ and $w'(u) = w(u)$ for all other u in G, where $h = w_{G_1}^C(x) + w_{G_1}^C(y) - w_{G1}^C\{x,y\} \geq 0$ and $g \geq h$. This is illustrated in Fig. 12.

(b) x is connected to y in G_2 through a degree 2 vertex x_1. Let $w'(x) = w^C_{G_1}(x) + g$, $w'(x_1) = w(x_1) + g$, $w'(y) = w^C_{G_1}(y) + g$ and $w'(u) = w(u)$ for all other u in G', where $g = w^C_{G_1}\{x,y\} - w^C_{G_1}(x) - w^C_{G_1}(y) \geq 0$.

Then the weight of a MWIS of G is the sum of $w(P_{G_1\backslash C'}) - g$ and $w'(P_{G'})$, where $P_{G'}$ is a MWIS of G' relative to w' (as shown in Figure 6).

Figure. 6. The weight modification for 2–separations

3. 2–amalgam split

This corresponds to the case that both C^1 and C^2 are partial joins, and in addition, there is a clique Q each of whose vertices are adjacent to all vertices involved in the two partial joins C^1 and C^2. Note that, in the amalgam split, C^2 is empty. This operation was proposed by Cornuejols and Cunningham [1986], and they also gave an $O(|V|^2|E|^2)$ algorithm for recognizing such a structure in general graphs.

4.2.2 The P_4–structure

1. Partner Decomposition

Theorem 4.2.6 (Even decomposition, Chvátal and Hoang [1985]). Suppose V(G) can be partitioned into V_1 and V_2 such that each induced P_4 has an even number of vertices in each V_i, $i = 1,2$, then G is perfect if and only if each of the subgraphs induced on V_1 and V_2 is perfect.

Theorem 4.2.7. (Odd Decomposition, Hoang [1985]). Suppose V(G) can be partitioned into V_1 (colored red) and V_2 (colored white) such that

(i) no induced P_4 has precisely two vertices of each color,

(ii) if an induced P_4 has precisely three vertices of one color, then at least one of these three vertices belongs to no monochromatic induced P_4.

Then G is perfect if and only if each of the subgraphs induced on V_1 and V_2 is perfect.

Two vertices x and y in G are said to be ***partners*** if there is a set S of three vertices such that both $S \cup \{x\}$ and $S \cup \{y\}$ induce a P_4. If the hypothesis of Theorem 4.2.1 or Theorem 4.2.1 is satisfied then obviously any two partners belong to the same V_i. Hence the following result implies both Theorem 4.2.6 and Theorem 4.2.7.

Theorem 4.2.8. Suppose V(G) can be partitioned into V_1 (colored red) and V_2 (colored white) so that all partners have the same color. Then G is perfect if and only if each of the subgraphs induced on V_1 and V_2 is perfect.

Consider the partner graph of G that has the same vertices as G, with any two vertices adjacent if and only if they are partners in G. Such a graph can be constructed in $O(n^5)$ steps. Then G has a nontrivial partition satisfying the hypothesis of Theorem 4.2.8 if and only if the partner graph of G is disconnected. Based on the even decomposition, bipartite graphs can be decomposed into two stable sets. Finally, the partner decomposition is self–complementary.

2. Forbidden P_4–types

Denote a P_4 by abcd. A 2–color assignment of vertices in P_4 can be represented by the following form: for example, RWWW, which stands for the assignment of red to a and white to b, c and d. The hypothesis of Theorem 4.3.1 "every two partners have the same color" can be replaced by a stronger hypothesis like "there is no P_4 of type RWWW or WRWW or WRRR or RWRR". Chvátal, Lenhart and Sbihi [1987] proved that there are precisely twelve theorems of this form:

Theorem 4.2.9. Let the vertices of G be 2–colored. If there is no P_4 of a type that belongs to S, then G is perfect if and only if each of each of its two subgraphs induced vertices of one color is perfect.

Six of this theorems arise by setting

S = {RWWW,WRWW,WRRR,RWRR},

S = {RRRR,WRRW,RWWR,WWWW}

S = {WRRR,WRRW,RWWR,WWWW,RWRR}

S = {WRRR,WRRW,RWWR,WWWW,RWRW}

S = {WRRR,WRRW,RWWR,RWWW,RWRW}

S = {WRRR,WRRW,WRWW}

and the remaining six theorems arise from these six by substituting $\overline{G}$ for G. Note that the first of these twelve theorems is exactly Theorem 4.2.2.

4.2.3 Graph Reduction

In this section we describe operations which do not necessarily yield perfection–preserving composition and decompositions, but they are useful in other aspects. To describe the first two operations, we need the following notations. A vertex x is said to be ***evenly*** (respectively, ***oddly***) ***related*** to y in G, denoted xE_Gy (respectively, xO_Gy) if every induced path connecting x to y in G has an ***even*** (respectively, ***odd***) number of edges.

1. **Even Merging**

Let u^1, u^2 be two evenly related vertices in G. Define the even merging of G relative to u^1 and u^2 to be the operation which replaces u^1 and u^2 by a single vertex u and makes u adjacent to every vertex in $N(u^1) \cup N(u^2)$. It was shown (Fonlupt and Uhry [1986]) that even merging on two evenly related vertices preserves perfection.

2. Odd Merging

Let u^1, u^2 be two oddly related vertices in G. Define the odd merging of G relative to u^1 and u^2 to be the operation which deletes u^1, u^2 and makes every other vertex in $N(u^1)$ adjacent to every one in $N(u^2)$. Denote the resulting graph by $\tilde{G}$. Hsu [1987a] showed the following:

Theorem 4.2.10. If G is perfect and either

(i) $(u^1,u^2) \notin E(G)$ and $u^1 O_G u^2$

or (ii) $(u^1,u^2) \in E(G)$ and (u^1,u^2) is not contained in any triangle,

then the odd merge $\overset{\approx}{G}$ relative to u^1 and u^2 is also perfect.

3. **Stable cutset** (Tucker [1983])

Let S be a stable cutset of G. Let H_i, $i = 1,\ldots,k$, be the connected components of $G \backslash S$. Let G_i be the subgraph of G induced on $H_i \cup S$, $i = 1,\ldots,k$. Tucker showed that

Theorem 4.2.11. G is a graph with a stable cutset S such that no odd hole of G contains a vertex of S, then G is r–colorable is and only if each of the H_i is r–colorable.

To simplify our discussion, we assume G is decomposed by the cutset S into two subgraphs G_1 and G_2. Given that both G_1 and G_2 are r–colored, Tucker's idea is to fuse together the corresponding vertices of S in G_1 and G_2 one at a time and, if the colors of the current pair x_1 (colored i), x_2 (colored j) being fused into x are not the same, then one can perform an i–j interchange to make them the same color.

4. Coloring K_4–free perfect graphs

Tucker [1987b] gave an $O(n^3)$ algorithm to 3–color K_4–free perfect graphs. The idea is to reduce this problem to that of coloring $(K_4\backslash e)$–free perfect graphs by doing K_3–contraction iteratively. Let $T_1 = wxy$ and $T_2 = xyz$ be two triangles which share an edge (x,y) and w, z are not adjacent. Define a ***K_3–contraction*** to be the operation that collapses w and z into one vertex, say w', and makes it adjacent to all vertices which were adjacent to either w or z.

Because it is possible to create odd holes after a K_3–contraction, Tucker showed that, whenever this happens, one can remove such odd holes by locating a star–cutset and separating the contracted graph into components. Each of these components can be iteratively contracted so that the final components are $(K_4\backslash e)$–free and easily colorable (Tucker [1987a]). These component colorings can then be combined to form a coloring of the original graph.

5. Decomposing $(K_4\backslash e)$–free perfect graphs

A $(K_4\backslash e)$–free graph G has at most $O(|E(G)|)$ maximal cliques (each edge of G is in at most one maximal clique). Fonlupt and Zemirline [1987] used the following three decomposition operations to separate a perfect $(K_4\backslash e)$–free graph into components that are either bipartite graphs or line graphs of bipartite graphs:

(1) a clique cutset.

(2) a stable cutset of size two.

(3) a separating structure consisting of a vertex z and the edges of all maximal cliques containing z, where z belongs to at least three maximal cliques, one at

least of size at least three.

The separating structure in (3) can be interpreted as a clique cutset in another graph G_q as defined below. Let the ***clique graph*** G_q of G be one in which $V(G_q)$ correspond to the set of maximal cliques of G, and two maximal cliques of G are adjacent in G_q if and only if they share a vertex. Note that even if G is perfect, G_q could still contain an odd hole. It can be shown that if G contains no twins, then there is a one–to–one correspondence between vertices in G and maximal cliques in G_q and vice versa (the definition of G_q implies a one–to–one correspondence between maximal cliques in G and vertices in G_q) and the separating structure in (3) corresponds to a clique cutset in G_q. Hence, the main theorem in Fonlupt and Zemirline [1987] can be restated as follows.

Theorem 4.2.12. Let G be a perfect $(K_4\backslash e)$–free graph. Then one of the following holds:

(i) G is bipartite or the line graph of a bipartite graph;

(ii) G has a clique cutset;

(iii) G_q has a clique cutset;

(iv) G has a stable cutset of size 2.

This theorem suggests that, in decomposing perfect graphs, one might consider decomposing some related graphs which could simplify the overall description.

6. **Frail composition** (Chvátal [1985])

An operation is called ***frail*** if it transforms graphs G_1 and G_2 into a graph G with the following property:

(4.2.13) If an induced subgraph H of G is an induced subgraph of neither G_1 nor G_2, then H or $\bar{H}$ has a star–cutset or else H has at most two vertices.

Denote the class of all graphs with at most two vertices by TRIV. We have

Theorem 4.2.14. Let $\mathscr{C}$ be any class of graphs such that TRIV $\subseteq \mathscr{C}$, and such that $\mathscr{C}$ is closed under taking induced subgraphs. Then $\mathscr{C}^*$ is closed under taking induced subgraphs and under all frail operations.

4.3. Combinatorial Algorithms

The algorithms discussed in this section can be applied to special classes of perfect graphs. They have the following features: (i) they make use of the graphical structures of the special class; (ii) the optimization algorithm often provides a proof that the class of graphs in consideration is either perfect or satisfies the SPGC; (iii) they are more efficient.

1. Triangulated graphs

Based on the property that every induced subgraph of a triangulated graph G contains a simplicial vertex, Fulkerson and Gross [1965] suggested the following iterative procedure to recognize triangulated graphs. Repeatedly locate a simplicial vertex and eliminate it from the graph. If, at some stage no simplicial vertex exists, then the graph is not triangulated. Otherwise, the final sequence of vertices obtained in this order is called a ***perfect elimination scheme*** (or PE ordering) of G. The existence of such PE schemes actually characterizes triangulated graphs. By constructing such schemes backwards, Rose, Tarjan and Leuker [1976] were able to give an $O(|V| + |E|)$ algorithm for recognizing triangulated graphs using lexicographic breadth–first–search (Lex BFS).

Let σ be a perfect elimination ordering for G. Fulkerson and Gross [1965] pointed out that every maximal clique was of the form $\{v\} \cup A_v$, where

$$A_v = \{ x \in N(v) \mid \sigma^{-1}(v) < \sigma^{-1}(x) \}.$$

Based on this, it is easy to find a maximum clique and the chromatic number of G in linear time. A maximum independent set (and a minimum clique cover) of G can be found by applying the following greedy algorithm to a PE ordering $v_1, \ldots, v_n$ of G (Gavril [1972]): Let $y_1 = v_1$. Recursively, once y_i is chosen, choose y_{i+1} to be the first available vertex in the sequence which is not adjacent to any of $y_1, \ldots, y_i$. Then the final set of y_i obtained, $\{y_1,\ldots,y_k\}$, is a maximum stable set and the collection of cliques of the form $\{y_i\} \cup A_{y_i}$ is a minimum clique cover.

2. Comparability Graphs

Comparability graphs admit transitive orientations. Transitivity induces an equivalence relation on the set of edges in E. A characterization of uniquely transitively orientable graphs was given by Shevin and Filippov [1970] and Trotter, Moore and Sumner [1976]. Based on the edge equivalence class, Golumbic gave an $O(\delta \cdot |E|)$ G–decomposition algorithm for recognizing comparability graphs and for calculating the exact number of transitive orientations, where δ is the maximum degree of G. Using the substitution decomposition, Spinrad [1985] gave another recognition algorithm, which will be discussed in Section 4.2.

A height function h on V(G) can be defined as follows. $h(v) = 0$ if v is a sink (i.e. no edge directed out from v); otherwise, $h(v) = 1 + \max \{ h(w) | v \to w \}$. This function can be assigned in linear time using a recursive depth–first–search. It gives $\omega(G)$ (the largest h(v)) and produces a minimum coloring for G (all v with the same h(v) receives the same color).

To find $\alpha(G)$, one can transform a transitive orientation into a network by adding two new vertices s and t and edges $s \to x$, $y \to t$ for each source x (no edge directed in) and sink y. Assigning a lower capacity of 1 to each vertex and call a minimum flow algorithm. The value of this flow is $\alpha(G)$ based on the min–flow max–cut theorem.

3. Interval graphs

Since interval graphs are also triangulated, the four optimization problems can all be solved using the algorithms for triangulated graphs. Based on the characterization that G is an interval graph if and only if G is a triangulated graph and $\overline{G}$ is a comparability graph, one can obtain an recognition algorithm for interval graphs based on those algorithms for triangulated graphs and comparability graphs. However, a linear time algorithm was obtained by Booth and Leuker [1976] based on the construction of PQ–trees. They made use of the following property (by Gilmore and Hoffman [1979]):

(4.3.1) a graph G is an interval graph if and only if its maximal cliques can be linearly ordered into $Q_1, \ldots, Q_k$ such that, for every vertex x, the maximal cliques containing x occur consecutively.

Given a graph G, they determined all maximal cliques (at most O(n) of them) and test if this property can be satisfied by starting with a single maximal clique and iteratively including one more maximal clique at a time. They stored the maximal cliques in a PQ–tree, which keeps track of all the potential linear orders satisfying () and adjusted this tree iteratively.

5. Meyniel graphs

Burlet and Fonlupt [1984] showed that Meyniel graphs can be built from basic Meyniel graphs by repeated applications of the amalgam operations. By extending the amalgamation to include substitution and complementation, we showed that Meyniel graph can be built from bipartite graphs and cliques. Conceivably, such a decomposition scheme can lead to polynomial optimization algorithms.

4. Claw–free perfect graphs

Minty [1980] and Sbihi [1981] gave polynomial algorithms for the maximum stable set problem on claw–free graphs. Minty reduced the problem of finding vertex–augmenting paths in a claw–free graph to that of finding edge–augmenting paths and used Edmond's matching algorithm. Sbihi, instead, developed a different type of blossom algorithm shrinking odd holes and cliques.

The other three problems of finding $\omega(G)$, $\gamma(G)$ and $\theta(G)$ turn out to be NP–hard (Hsu [1980]) in general claw–free graphs. Hsu and Nemhauser [1981,1982] and Hsu [1980,1981] gave polynomial algorithms for finding these parameters on their perfect subclass. A common feature of these algorithms is that they are all related to bipartite matching.

Chvátal and Sbihi [1988] gave a polynomial algorithm to recognize claw–free perfect graphs. They decompose a given graph based on clique cutsets into components belonging to certain classes of primitive graphs, one of which is a subclass of Gallai–perfect graphs they referred to as elementary graphs.

6. Planar perfect graphs

Tucker's [1973] proof that the SPGC holds for planar graphs actually yields a coloring algorithm for their perfect subclass. Later, Tucker and Wilson [1984] improved its complexity to $O(n^2)$. Hsu [1986] designed another coloring algorithm based on simple decompositions which produce components that are "almost" uniquely colorable. However, these two approaches only work on unweighted case.

Motivated by the coloring algorithm of Hsu [1986], We discovered a decomposition recognition algorithm for planar perfect graphs (Hsu [1987b]). The primitive classes of this decomposition are (i) planar comparability graphs; (ii) planar line graphs of planar bipartite graphs; and (iii) a small class of 10 exceptional graphs which can all be generated from a typical configuration (as shown in Figure 7). Because of the simplicity of edge connections in planar graphs, our separation structures can be classified by the number of vertices contained in the cutsets. We use four kinds of cutsets. A 1–cutset is an articulation vertex. A 2–cutset is either a 2–clique or a stable set of size 2. A three cutset is a set of three vertices which is not a stable set. These cutsets have all been covered in Section 4.2. Finally, a 4–cutset is an induced 4–cycle.

Figure 7. A special class of planar perfect graphs

These decomposition schemes can be used to solve all four optimization problems on planar perfect graphs (Hsu [1988]). Since planarity is not needed in the above implementation, these methods can actually be applied to any perfect graphs whose inseparable components under these four separation structures have polynomial algorithms for the four optimization problems.

7. $(K_4\backslash e)$–free (diamond–free) perfect graphs

Every diamond–free graph satisfies the property that for each vertex v, N(v) consists of a set of cliques among which no edge connects vertices in different cliques. Hence, there is a linear number of maximal cliques in G,and $\alpha(G)$ is easy to compute. Tucker [1987] showed that every diamond–free perfect graph G contains a vertex v such that at most two cliques in N(v) have size greater than 1. His method is as follows. Starting from any given vertex v_1, one can build a maximal chordless path $P = v_1v_2...v_k$ where edge (v_i,v_{i+1}) is in clique C_i and for $i+1 < j$, cliques C_i and C_j are disjoint. Then v_k must be a desired vertex. This result immediately yields a coloring algorithm: Suppose $G\backslash\{v\}$ has already been colored using $\omega(G)$ colors. Let C_1, C_2 be two cliques in N(v) of size greater than 1. Consider a color i not used in C_1 and a color j not used in C_2. Then the set S of vertices in N(v) with colors i or j form an independent set. Let H be the subgraph induced on all vertices colored i or j. It is easy to see that no i–vertex in S can be in a component of H containing any j–vertex in S. Hence, one can switch the color of all j–vertices of S into i and assign the color j to vertex v.

Fonlupt and Zemirline [1987] recently discovered a decomposition recognition algorithm for diamond–free perfect graphs as described in Section. We believe that optimization algorithms using that decomposition scheme can eventually be designed.

6. K_4–free (3–chromatic) perfect graphs

An optimal coloring algorithm for this class was recently designed by Tucker [1987]. The main idea is to reduce the coloring problem to that of $(K_4\backslash e)$–free perfect graphs. To achieve this, Tucker eliminates each diamond of G by identifying its two nonadjacent vertices, and decomposing the resulting graph in case odd holes are created. We briefly describe his approach as follows. Let $T_1 = x_1yz$, $T_2 = x_2yz$ be two triangles in a $(K_4\backslash e)$–free perfect graph G such that x_1 and x_2 are not adjacent. Identify x_1 with x_2 and let x be the resulting vertex, which is adjacent to all vertices that were adjacent to either x_1 or x_2. Let G' be the resulting graph and T be the triangle xyz in G'. Tucker showed that if G' contains an odd hole C', then

(a) C' contains exactly one vertex in T, say x,

(b) the other two vertices of T each form one or more triangles in G' with vertices of C'.

Then at least one of $\{y\} \cup N(y)$ and $\{z\} \cup N(z)$ is a star–cutset in G' and we can decompose G with respect to any star–cutset until no star–cutset exists in any components. Furthermore, G can be 3–colored if and only if each of the components can be 3–colored.

5. Related Problems and Applications

We shall discuss problems and techniques derived from the study of perfect graphs in the following three areas: 1. decomposition; 2. recognition; 3. optimization.

Many special classes of perfect graphs can be recognized by applying suitable decomposition schemes. Most of these can be used to solve the corresponding optimization problems provided that good characterizations of the inseparable components are available. Some of these decompositions can also be applied to graphs that are not necessarily perfect.

1. Kernel–solvability

Define a kernel of a digraph G to be a subset of vertices $K \subseteq V$ which is both independent and absorbing (every vertex in $V \backslash K$ has a successor in K). When every induced subgraph of G has a kernel, the digraph G is said to be kernel–perfect (Duchet [1980]). A reversible arc $x \to y$ is one such that $y \to x$ also exist. A subdigraph of G is said to be complete if its vertices are pairwise adjacent (its underlying undirected graph). An orientation of a graph is ***normal*** if every complete induced subgraph of G has a kernel. An undirected graph G is said to be ***(kernel)–solvable*** if every normal orientation of G is kernel–perfect. Berge and Duchet [1983] conjectured that

(5.1.1) **A graph is perfect if and only if it is (kernel)–solvable.**

A sufficient condition (Galenan–Scnchez and Neumann–Lara [1984]) for a digraph to be kernel perfect is the following

(5.1.2) Every odd directed cycle C has two chords whose terminal endpoints are consecutive on C.

A special type of normal orientation is called an ***M–orientation*** if every directed triangle possesses at least two reversible arcs. An undirected graph G is *(kernel)–M–solvable* if every M–orientation of G is kernel–perfect. Thus, a weaker form of (5.1.1) is

(5.1.3) Every perfect graph is (kernel)–M–solvable.

Duchet [1987] showed that parity graphs are M–solvable.

2. The recognition of circle graphs and circular–arc graphs

The join decomposition has been used by Gabor, Hsu and Supowit [1985] to recognize circle graphs. They showed that

(i) A graph G is a circle graph iff every j–inseparable component of G is a circle graph.

(ii) Each j–inseparable circle graph has a unique chord model.

They also gave an $O(m \cdot n)$ algorithm to construct a chord model for a j–inseparable circle graph. Such a decomposition scheme can also be used to solve the isomorphism problem trivially.

Hsu [1987c] reduced the recognition problem of circular–arc graphs to that of circle graphs based on a definition of normalized representations.

3. Covering Orthogonal Polygons

The *art gallery problem* is to determine a minimum number of guards in a polygon such that they see every point of the polygon. This problem is NP–hard (O'Rourke and Supowit). It can be restated as the problem of covering a polygon with a minimum number of star polygons (a star polygon contains a point that sees every point of the polygon). An *orthogonal polygon* is one with all its sides parallel to one of the two coordinate axis. Define an independent set of points in P with respect to a class of covering polygons C, denotes a set of points in P, no two of which can be covered by any polygon in C. Consider the following duality theorem:

(5.3.1) The size of a minimum cover by polygons from class C is equal to the size of a maximum independent set of points with respect to the class C. Chaiken et al [1981] first showed that this theorem holds for polygons that are orthogonally convex (OCP). Gy'ori [1984] then showed that it holds if the polygon is only vertically convex. Later, Saks [1982] showed that the graph determined by the boundary squares of the grid induced by the vertices of an OCP is perfect. This theorem has now been shown to hold for covering orthogonal polygons with orthogonally star polygons (Motwani et al. [1988]). Their approach is to study the visibility graphs arisen from certain special orthogonal polygons. Some of these turn out to be perfect, e.g. permutation graphs, weakly triangulated graphs and available optimization algorithms (e.g. minimum clique cover) can be readily applied.

4. The odd hole recognition problem

The recognition problems for a number of special classes of perfect graphs such as planar perfect graphs, $(K_4\backslash e)$–free perfect graphs and claw–free perfect graphs are actually equivalent to the **odd hole recognition (OHR)** problems on planar graphs, $(K_4\backslash e)$–free graphs and claw–free graphs, respectively, because the SPGC holds for these classes and they do not have odd antiholes of size at least seven, by definition. The status of the OHR problem on general graphs remains open. Suppose the SPGC is true, then a polynomial OHR problem can be used to recognize perfect graphs.

One problem related to the OHR is the ***path–parity*** problem: given two vertices u and v in G, determined if there exist two induced paths connecting them with different parity. Its status is also open. Another problem is to test whether two given vertices u and v in G are contained in any hole. This can be solved for planar graphs (Hsu [1987d]). An understanding of this problem is very useful in designing decomposition schemes for perfect graphs.

First of all, the family of maximal cliques in a $(K_4\backslash e)$–free graph G satisfies the Helly property: any subfamily F of maximal cliques in which any two cliques share at least one vertex has a common vertex in all maximal cliques in F. Hence a maximal clique $F = \{Q_1,...,Q_k\}$ in G_q satisfies that all the Q_i's contain a common vertex z. If they contain two common vertices u and v, then u, v must be contained in every clique of G, and form a twin. Therefore, there is a unique common vertex contained in all the cliques in F. On the other hand, for each vertex z of G, the set F of maximal cliques in G containing z gives rise to a maximal clique in G_q. Hence

Now, consider the separating structure in (3). Let F_G be the set of maximal cliques in G; let F_z be the set of maximal cliques containing z. Deleting vertex z and edges of all maximal cliques containing z in G reduce F_G to $F_G\backslash F_z$. Since F_z gives rise to a maximal clique in G_q, this operation corresponds to deleting all vertices of the maximal clique F_z in G_q. This verifies that the separating structure in (3) corresponds to a clique cutset in G_q. The restriction placed on z amounts to eliminating trivial cases: if there are only two maximal cliques in G containing z, there is no need to decompose G at z; if all maximal cliques containing z are of size 2, then {z} gives rise to a clique cutset in G, which can be covered by (1).

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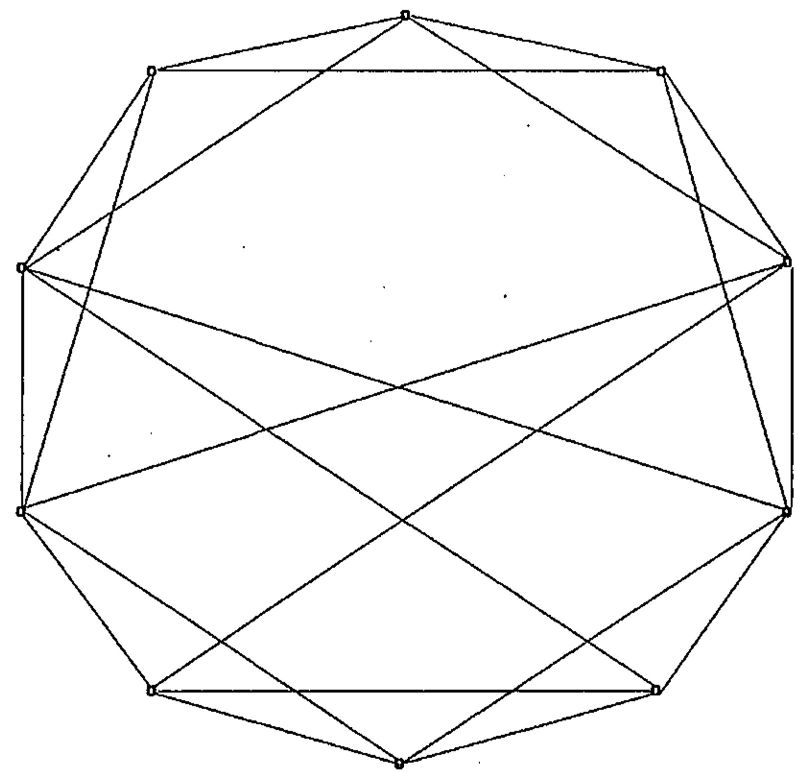


Figure 1. A graph G which is partitionable but not minimal imperfect

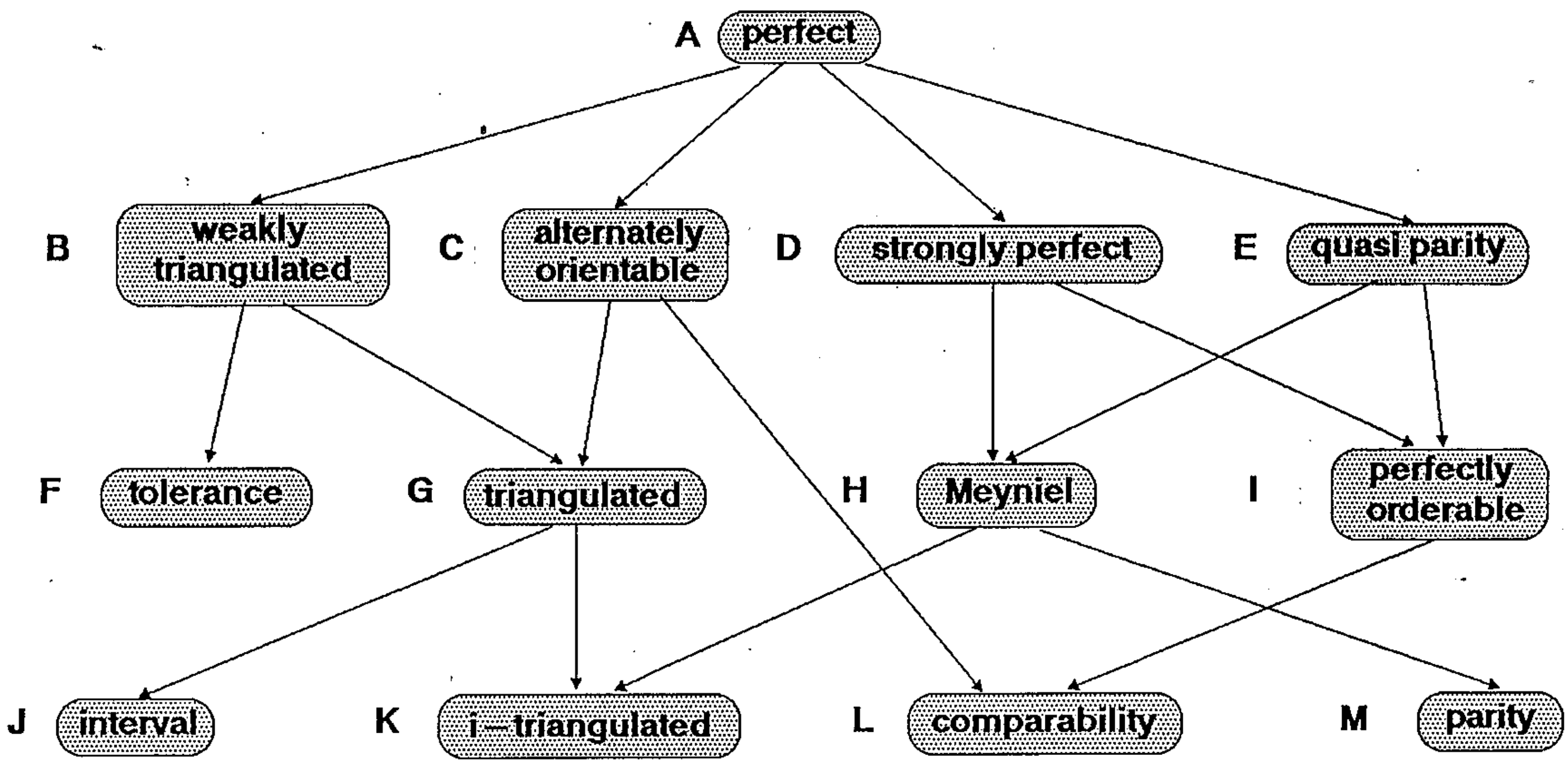


Figure 2. The Hasse diagram of the containment relationships among some classes of perfect graphs

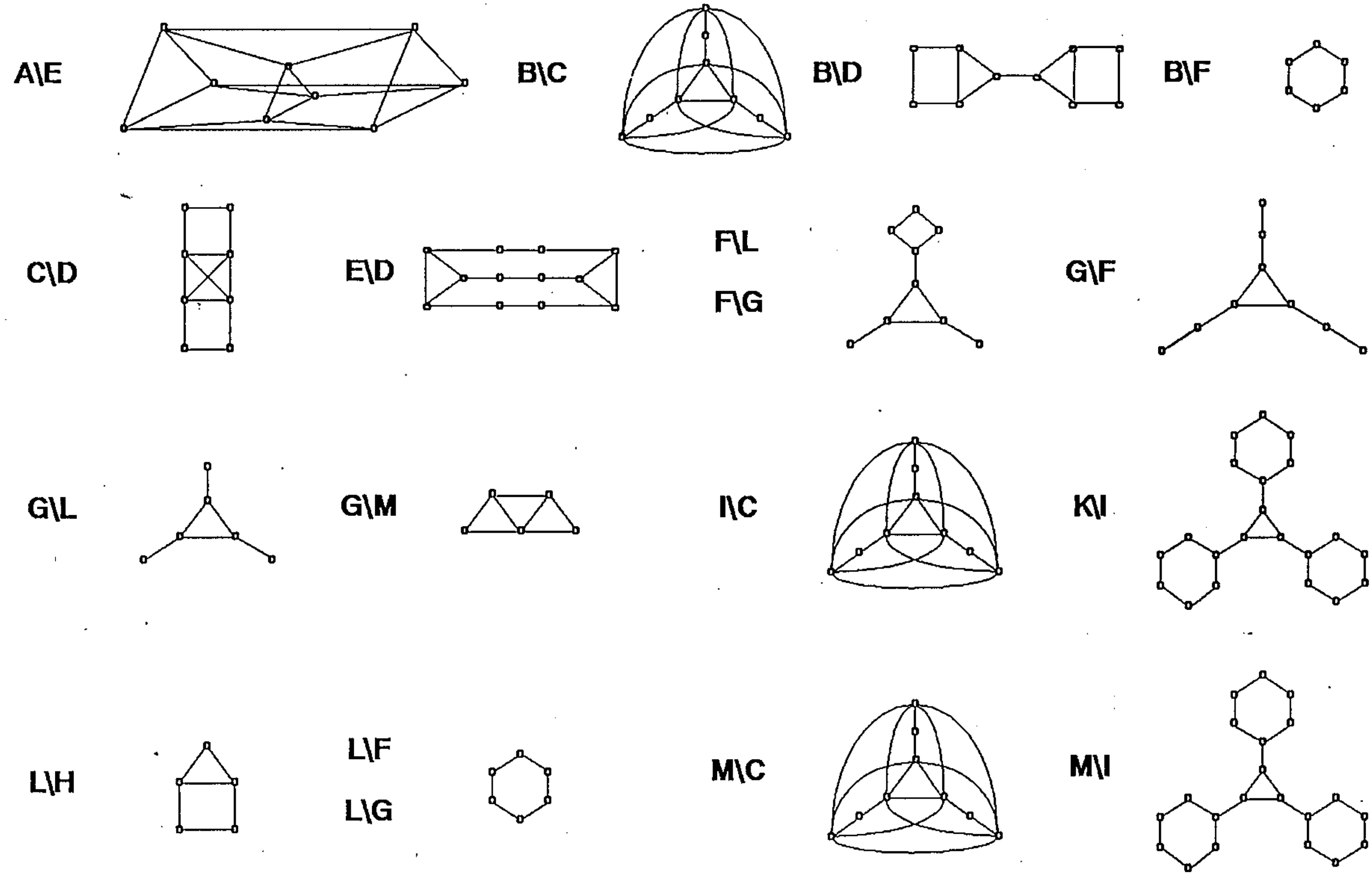


Figure 3. Examples showing the differences among these special classes

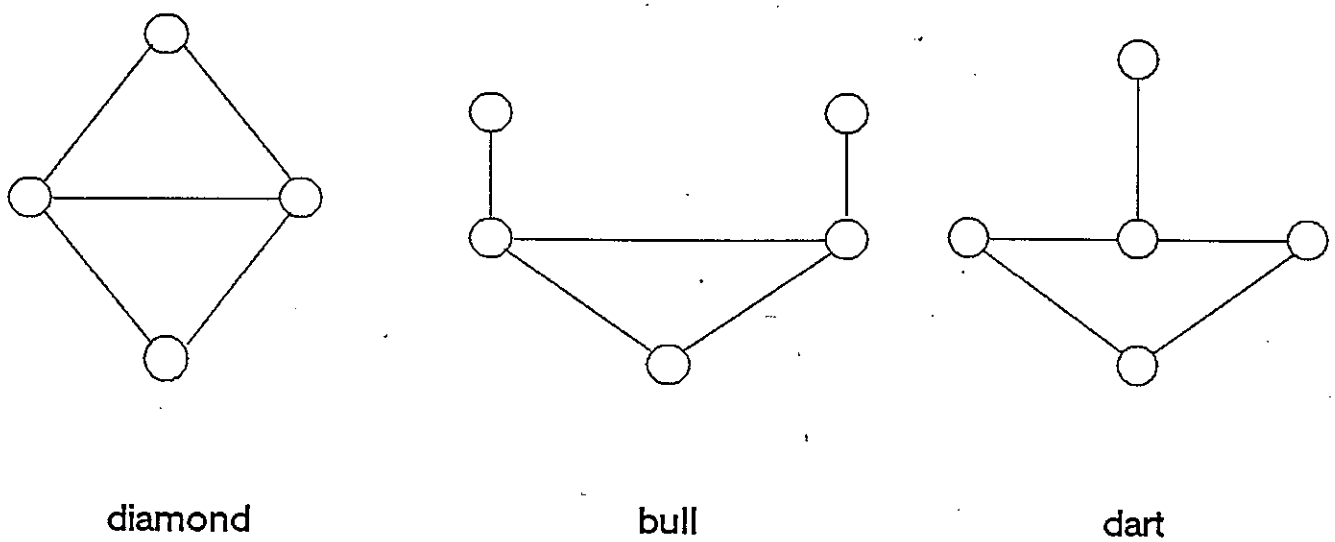


Figure 4. Three graphs: diamond, bull and dart

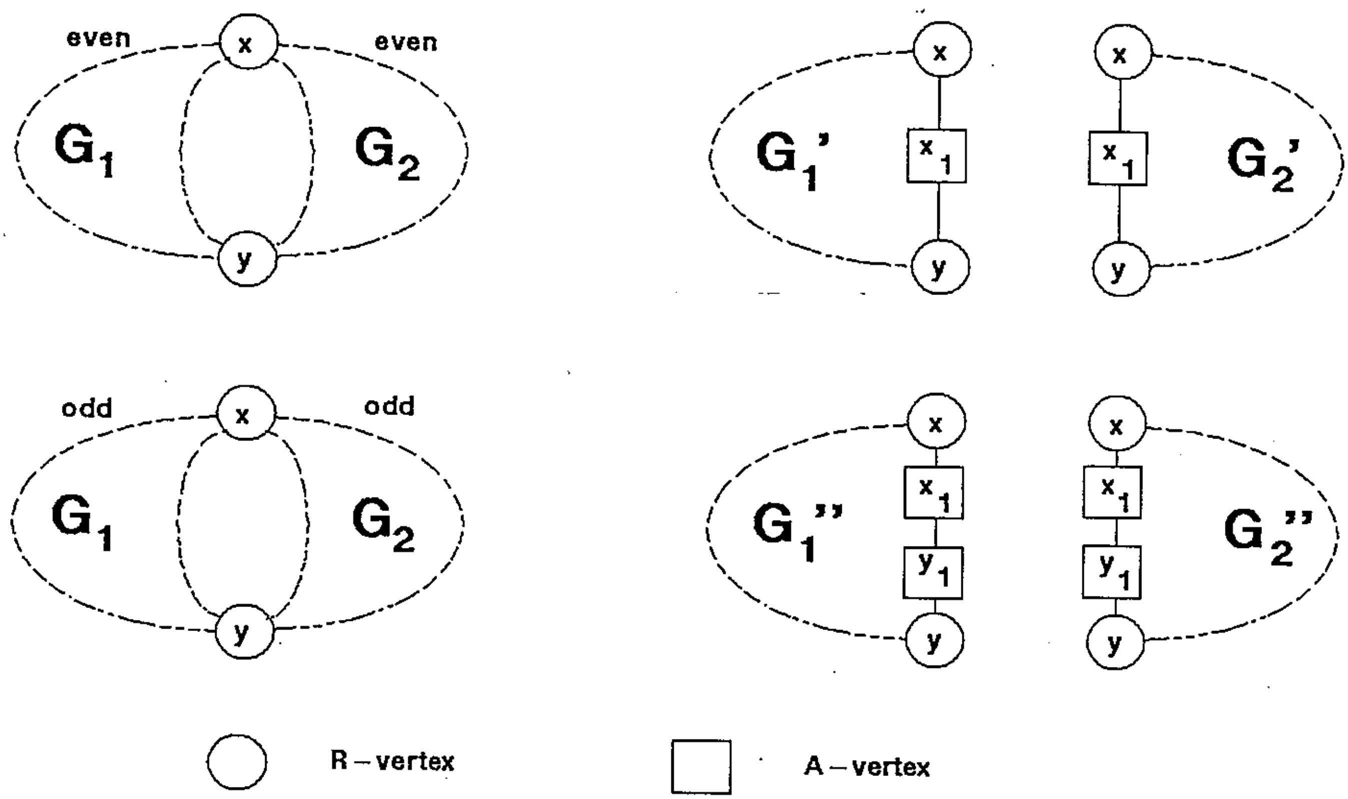


Figure 5. A case for 2 – separation

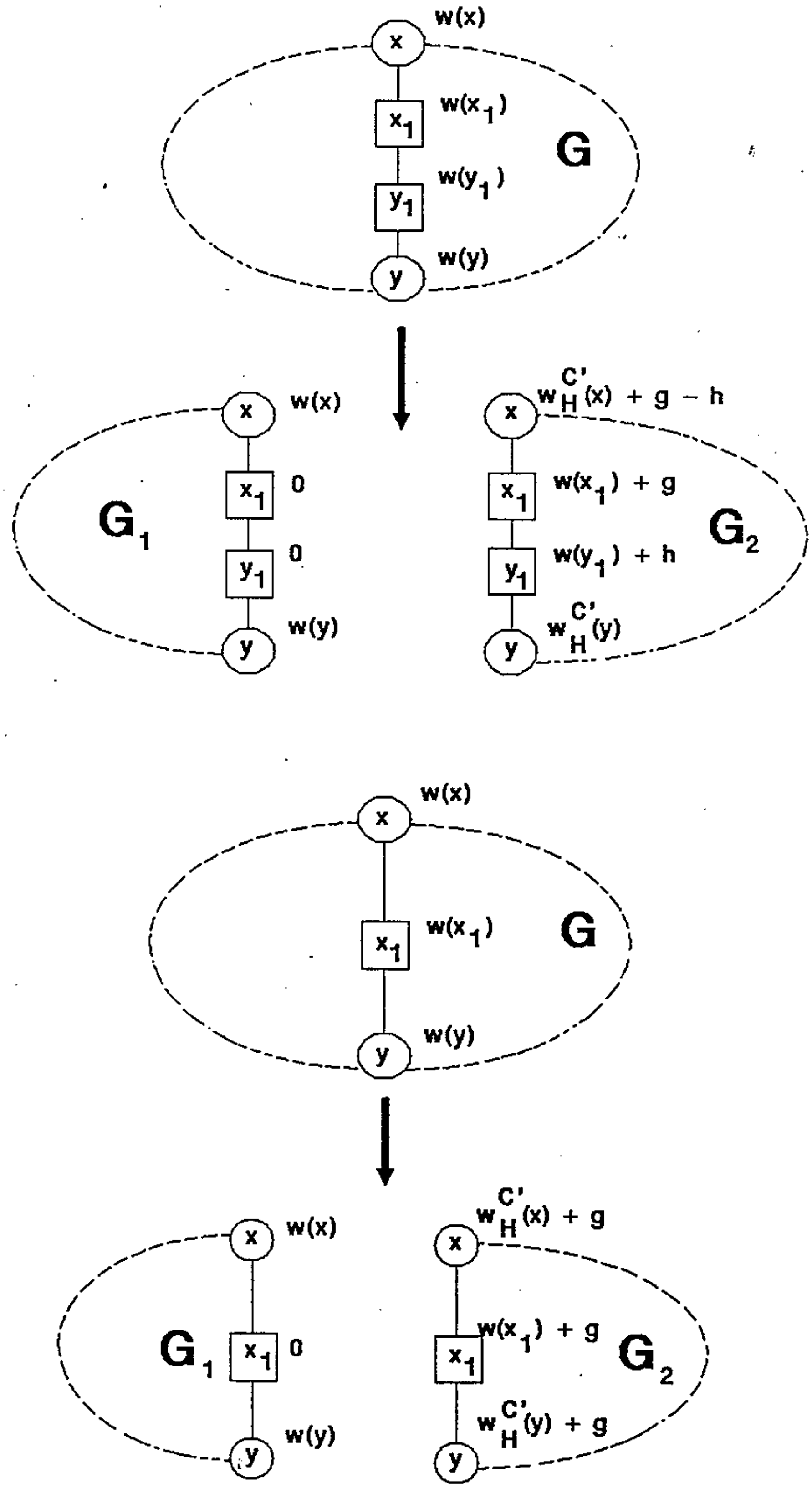
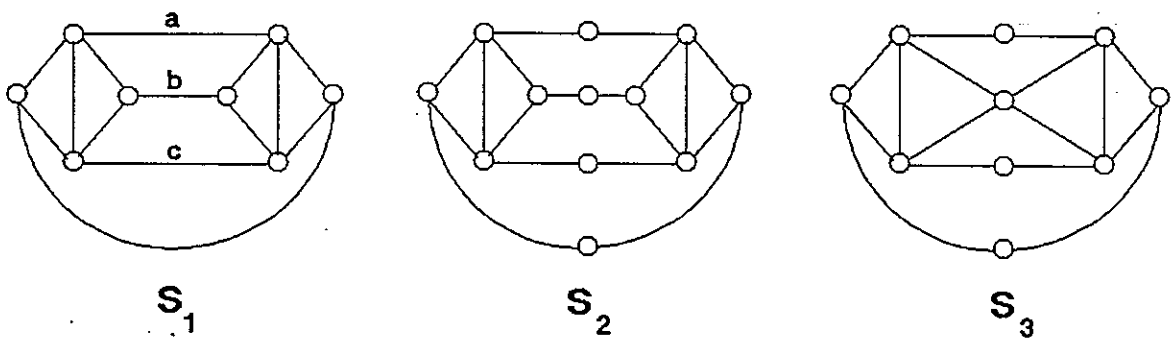


Figure 6. The weight modification for 2 – separations



Class S consists of S_1^*, S_2 and S_3, where S_1^* is the collection of graphs consisting of S_1 and any other graph obtained by replacing one or more edges in {a,b,c} of S_1 with induced paths of length 3

Figure 7. A special class of planar perfect graphs