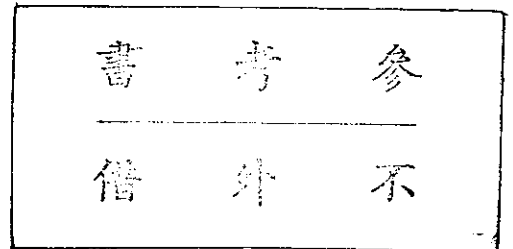# UNIFYING MAXIMUM CUT AND MINIMUM CUT

OF A PLANAR GRAPH

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Abstract

We consider the real-weight maximum cut of a planar graph: Given an undirected planar graph with real-valued weights associated with its edges, find a partition of the vertices into two nonempty sets such that the sum of the weights of the edges connecting the two sets is maximum. The conventional maximum cut and minimum cut problems assume nonnegative edge weights, and thus are special cases of the real-weight maximum cut. We develop an $O(n^{3/2} \log n)$ algorithm for finding a real-weight maximum cut of a planar graph where n is the number of vertices in the graph. The best maximum cut algorithm previously known for planar graphs has the running time of $O(n^3)$.

Index terms

algorithm

even-degree edge set

maximum cut

maximum weight matching

minimum cut

minimum cycle

planar graph

planar separator theorem

Figure captions

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Fig. 5. The star for vertex v in G'/v

1. Introduction

Conventionally, the maximum cut problem and the minimum cut problem for undirected graphs have been treated differently. In fact, for general graphs finding a maximum cut is NP-hard [6] while the minimum cut problem can be solved in polynomial time by finding minimum (s,t)-cuts or equivalently by finding maximum flows [8]. When the graph is restricted to be planar, the complexities of both problems are reduced: the maximum cut problem can be transformed into a maximum weight matching, thus is polynomial-time solvable [7] [1]; the minimum cut problem can be solved more efficiently since a minimum (s,t)-cut can be found by finding shortest paths [9] [14].

In this paper, we take a unified approach to the maximum cut and minimum cut problems for planar graphs. We consider the following real-weight maximum cut problem. Let $G = (V,E)$ be a connected undirected graph. Assume that each edge $e \in E$ has an associated real-valued weight $w(e)$. The real-weight maximum cut problem is to find a partition of V into two nonempty sets such that the sum of the weights of the edges connecting the two sets is maximum. It is apparent that the conventional maximum cut and minimum cut problems (with nonnegative edge weights) are special cases of the real-weight maximum cut problem. (A minimum cut of a graph corresponds to a maximum cut of the negated graph.)

Hadlock has demonstrated that for planar graphs the maximum cut problem can be reduced to finding a maximum weight matching for a

complete graph [7]. This approach thus yields an algorithm of time $O(n^3)$ where n is the number of vertices in the graph [4] [10]. On the other hand, a minimum cut of a planar graph can be found by finding $O(n)$ minimum (s,t)-cuts [8]. Thus the minimum cut problem has an upper bound of $O(n^2 \log^2 n)$ [14] [15]. It should be mentioned that none of these algorithms can be applied to the more general real-weight maximum cut and we are not aware of any algorithm that can.

We investigate the real-weight maximum cut problem via a classic result in the planar graph theory: A connected graph $G = (V,E)$ is planar iff it has a (combinatorial) dual $G_d = (V_d, E_d)$, i.e. there is a one-to-one correspondence $f:E \longrightarrow E_d$ which maps a minimal cut of G to a simple cycle of G_d and vice versa [3]. We shall generalize this well-known result to establish a one-to-one correspondence between the cuts of G and the even-degree edge sets of G_d. Then finding a maximum cut of G is equivalent to finding a maximum (weight) even-degree edge set of G_d.

We propose an algorithm for finding a maximum even-degree edge set of G_d. The algorithm consists of two parts: If there is a positive cycle in G_d, then the problem can be reduced to finding a maximum weight matching of a sparse graph. By applying Lipton and Tarjan's planar graph separator theorem [11] [12], such a maximum weight matching can be found in $O(n^{3/2} \log n)$ time [13]. On the other hand, if G_d has no positive cycle, then a maximum even-degree edge set of G_d corresponds to a minimum cycle in the negated graph G_d- of G_d where G_d- contains no negative cycle. An $O(n^{3/2} \log n)$ algorithm has been presented for

detecting a minimum cycle passing through a specified vertex in a planar graph [13]. We shall modify this algorithm for detecting a minimum cycle in G_d in the same time complexity. Consequently, we have an $O(n^{3/2} \log n)$ algorithm for finding a maximum even-degree edge set of G_d. In other words, a real-weight maximum cut of a planar graph can be found in $O(n^{3/2} \log n)$ time.

In the next section, we introduce basic definitions and demonstrate the correspondence between cuts and even-degree edge sets. In Section 3, a planar graph is triangulated so that its dual becomes a cubic planar graph. Then we characterize a maximum even-degree edge set in such a cubic graph. The reductions that relate a maximum even-degree edge set to a maximum weight matching and the algorithms will be described in Section 4. Finally, we make some concluding remarks in the last section.

2. Cuts and even-degree edge sets

In this paper, all graphs and multigraphs are undirected. Multigraphs can have self-loops and parallel edges but graphs can not. Given a connected graph $G = (V,E)$, let $E(A,B)$ denote the set of edges of G that connect two disjoint vertex sets A and B. An edge set $C \subseteq E$ is a cut if there is a partition of V into two nonempty sets X and $\bar{X}$ $(=V-X)$ such that $C = E(X,\bar{X})$. A cut is minimal if none of its proper subsets is a cut.

Lemma 1. The union of two disjoint cuts is a cut.

Proof: Let $C = C_1 \cup C_2$ where $C_1 = E(X,\bar{X})$, $C_2 = E(Y,\bar{Y})$ and $C_1 \cap C_2 = \emptyset$. Consider the vertex sets $X \cap Y$, $X \cap \bar{Y}$, $\bar{X} \cap Y$ and $\bar{X} \cap \bar{Y}$ as shown in Fig. 1. Since $C_1 \cap C_2 = \emptyset$, we have

$$E(X \cap Y, \bar{X} \cap \bar{Y}) = E(X \cap \bar{Y}, \bar{X} \cap Y) = \emptyset ,$$

$$C_1 = E(X \cap Y, \bar{X} \cap Y) \cup E(X \cap \bar{Y}, \bar{X} \cap \bar{Y}) \text{ and}$$

$$C_2 = E(X \cap Y, X \cap \bar{Y}) \cup E(\bar{X} \cap Y, \bar{X} \cap \bar{Y}).$$

Thus $C = C_1 \cup C_2$ is a cut separating $(X \cap Y) \cup (\bar{X} \cap \bar{Y})$ and $(X \cap \bar{Y}) \cup (\bar{X} \cap Y)$.   Q.E.D.

Lemma 2. Let C_1 and C_2 be cuts of G and $C_1 \subseteq C_2$. Then $C = C_2 - C_1$ is also a cut of G.

Proof: let $C_2 = E(X,\bar{X})$ and $C_1 = E(Y,\bar{Y})$. Consider the vertex sets $X \cap Y$, $X \cap \bar{Y}$, $\bar{X} \cap Y$ and $\bar{X} \cap \bar{Y}$ as shown in Fig. 2. Then

$$C_2 = E(X \cap Y, \bar{X} \cap Y) \cup E(X \cap Y, \bar{X} \cap \bar{Y}) \cup E(X \cap \bar{Y}, \bar{X} \cap Y)$$
$$\cup E(X \cap \bar{Y}, \bar{X} \cap \bar{Y}).$$

Since $C_1 \subseteq C_2$ and C_1 is a cut separating Y and $\bar{Y}$, we have

$$C_1 = E(X \cap Y, \bar{X} \cap \bar{Y}) \cup E(X \cap \bar{Y}, \bar{X} \cap Y) \text{ and}$$

$$E(X \cap Y, X \cap \bar{Y}) = E(\bar{X} \cap Y, \bar{X} \cap \bar{Y}) = \phi \ .$$

Consequently, $C = C_2 - C_1 = E(X \cap Y, \bar{X} \cap Y) \cup E(X \cap \bar{Y}, \bar{X} \cap \bar{Y})$ and C is a cut separating $(X \cap Y) \cup (\bar{X} \cap \bar{Y})$ and $(X \cap \bar{Y}) \cup (\bar{X} \cap Y)$.   Q.E.D.

Applying Lemmas 1 and 2 repeatedly, one can decompose a cut into minimal cuts.

Lemma 3. An edge set $C \subseteq E$ is a cut of $G = (V,E)$ iff it is a union of disjoint minimal cuts of G.

Now we consider a connected multigraph $G_d = (V_d, E_d)$. A nonempty edge set $D \subseteq E_d$ is said to be even-degree if each vertex of V_d is incident to an even number of edges in D. Obviously a simple cycle is an even-degree edge set and none of its proper subset is even-degree. (A self-loop is a simple cycle.)

The even-degree edge sets have essentially the same properties as the cuts. We just state the result without proof.

Lemma 4. An edge set is even-degree iff it is a union of disjoint simple cycles.

In the planar graph theory, the minimal cuts of a planar graph are associated with the simple cycles of its dual graph [3]. The following lemma relates the cuts to the even-degree edge sets.

Lemma 5. Let $G = (V,E)$ be a connected graph and let $G_d = (V_d, E_d)$ be a

connected multigraph. Assume that $f: E \longrightarrow E_d$ is a one-to-one correspondence. Then the following two conditions are equivalent:

(1) For $C \subseteq E$, C is a minimal cut of G iff $f(C)$ is a simple cycle in G_d.

(2) For $C \subseteq E$, C is a cut of G iff $f(C)$ is an even-degree edge set of G_d.

Proof: (2) is implied by (1) due to Lemmas 3 and 4. Next we prove that (2) imples (1).

Assume that C is a minimal cut of G. Then $f(C)$ is an even-degree edge set of G_d by (2). If $f(C)$ is not a simple cycle, then $f(C)$ has a proper subset D which is a simple cycle. Consequently $f^{-1}(D)$ is a proper subset of C and $f^{-1}(D)$ is a cut of G by (2). This contradicts the fact that C is minimal. Thus $f(C)$ must be a simple cycle.

To prove the converse, let $f(C)$ be a simple cycle in G_d. Then C is a cut of G by (2). If C is not minimal, then C has a proper subset C_0 which is also a cut. Consequently $f(C_0)$ is a proper subset of $f(C)$ and is even-degree by (2). This contradicts the fact that $f(C)$ is a simple cycle. Thus C must be a minimal cut.   Q.E.D.

Note that (1) characterizes G_d as a combinatorial dual to G [3]. Thus we have the following theorem.

Theorem 1. A connected graph $G = (V,E)$ is planar iff there is a multigraph $G_d = (V_d, E_d)$ and a one-to-one correspondence $f: E \rightarrow E_d$ which maps a cut of G to an even-degree edge set of G_d and vice versa.

3. Regularizing the graph

Let $G = (V,E)$ be a connected planar graph with n $(= |V|)$ vertices. Assume that a real-valued weight is assigned to each edge. To find a (real-weight) maximum cut of G, we first triangulate G by adding some new edges. A triangulation $G_t = (V,E_t)$ of G is a connected planar graph satisfying

  (i) $E \subseteq E_t$,

  (ii) Each vertex of G_t has degree at least 2, and

  (iii) G_t can be embedded in the plane such that each face of G_t is

  enclosed by a simple cycle of three edges.

And we assign zero weight to each new edge in $E_t - E$.

Lemma 6. A maximum cut of $G = (V,E)$ corresponds to a maximum cut of $G_t = (V,E_t)$, and vice versa.

Note that G_t can be constructed from G in $O(n)$ time and G_t still has $O(n)$ edges as G does. Let $G_d = (V_d,E_d)$ be a dual of G_t. Then G_d can be constructed from G_t in $O(n)$ time and G_d is a cubic planar graph, i.e. each vertex of G_d has degree 3. We assign to each edge of E_d the same weight as its corresponding edge of E_t. Then due to Theorem 1, finding a maximum cut of G_t is equivalent to finding a maximum (weight) even-degree edge set of G_d.

In $G_d = (V_d,E_d)$, a cycle is said to be positive (negative, nonnegative) if its total weight is positive (negative, nonnegative). A cycle is minimum (maximum) if its total weight is minimum (maximum).

The following lemma characterizes a maximum even-degree edge set.

Lemma 7. Let D be a maximum even-degree edge set of G_d. If G_d contains a nonnegative cycle, then D is a union of vertex-disjoint nonnegative cycles. If G_d contains no nonnegative cycle, then D is a maximum cycle in G_d.

Proof: Since G_d is a cubic graph, each vertex of G_d is adjacent to 0 or 2 edges in D. Thus D is a union of vertex-disjoint cycles in G_d. The claim then follows from the fact that D is maximum.   Q.E.D.

4. Reductions and algorithms

In this section, we consider the problem of finding a maximum even-degree edge set in a real-weight cubic planar graph $G_d = (V_d, E_d)$. We first show that this problem can be reduced to finding a maximum weight matching provided that G_d contains a positive cycle.

A matching M of graph $G = (V, E)$ is a set of edges no two of which have a common vertex. If $|M| = |E|/2$, then M is called a complete matching. Assume that each edge of G has an associated real-valued weight. A maximum weight matching (minimum complete matching) is a matching (complete matching) of G whose total weight is maximum (minimum).

To find a maximum even-degree edge set of $G_d = (V_d, E_d)$, we construct a graph $G' = (V', E')$ from G_d. Each vertex v of G_d is replaced by a "star" in G' and each edge e of G_d has a surrogate in G' as depicted in Fig. 3 and 4. Define the edge weights of G' as follows: the surrogate of each edge $e \in E_d$ has the same weight as e; and all new edges in stars have zero weights. It is apparent that G' has $O(n)$ vertices and $O(n)$ edges and G' can be constructed from G_d in $O(n)$ time. Similar constructions have appeared in [16], [10], [13]. We have the following lemma.

Lemma 8. Let $M \subseteq E'$ be a minimum complete matching of $G' = (V', E')$. If $E_d - M \neq \emptyset$, then $E_d - M$ is a maximum even-degree edge set of G_d.

Proof: Let $M \subseteq E'$ be any complete matching of G' such that $E_d - M \neq \emptyset$.

If M does not contain edge (u',u") in a star substituting a vertex v of G_d (see Fig. 3), then M must contain (v',u'), (v",u") and all the edges incident to v in G_d and hence v has degree 0 in the subgraph of G_d induced by E_d - M. On the other hand, if M contains (u',u"), then v has degree 2 in the subgraph. Thus E_d - M is an even-degree edge set of G_d. Conversely, let D be any even-degree edge set of G_d. As shown in Lemma 7, D is a union of vertex-disjoint cycles in G_d. Thus from the construction of G', one can easily observe that there exists a complete matching M of G' such that D = E_d - M (see Fig. 4). Clearly the weight of E_d - M is maximum if and only if the weight of M is minimum.

<div align="right">Q.E.D.</div>

A minimum complete matching of G' can be found by finding a maximum weight matching of the same graph except that the weight w(e) of each edge e ∈ E' must be replaced by a new weight W - w(e) where W is a large constant [10]. Lipton and Tarjan have presented an $O(n^{3/2} \log n)$ algorithm for finding a maximum weight matching of a planar graph by applying the planar separator theorem [11] [12]. For graph G' = (V',E') which is not always planar, the same "divide-and-conquer" method can still be applied as K. Matsumoto, et al. have pointed out [13].

Lemma 9. A maximum weight matching of G' = (V',E') can be found in $O(n^{3/2} \log n)$ time.

Note that Lemma 9 has been demonstrated in [13] for a graph slightly different from G'. But the same deductions can be carried over for G'.

Directly from the proof of Lemma 8 is the following simple result: If G_d contains a positive cycle, then $E_d - M \neq \phi$ where M is a minimum complete matching of G'. Thus Lemmas 8 and 9 have suggested an efficient algorithm for the maximum even-degree edge set of G_d provided that G_d has a positive cycle. In the case that G_d contains no positive cycle, a maximum cycle in G_d is then a desired maximum even-degree edge set (Lemma 7). In the following, we assume that the weights associated with the edges of G_d have been negated. Thus we want to find a minimum cycle in graph $G_d = (V_d, E_d)$ where G_d contains no negative cycles.

Finding a minimum cycle in G_d can be reduced to finding maximum weight matchings in certain graphs augmented from G'. Let v be a specified vertex in G_d. Denote by $G'/v = (V'/v, E'/v)$ the graph augmented from G' by adding two vertices and two edges to the star substituting v in G' as shown in Fig. 5. (Other stars remain as in Fig. 3.) We assign zero weights to the two new edges.

Lemma 10. Assume that $M \subseteq E'/v$ is a complete matching of $G'/v = (V'/v, E'/v)$. Then M is maximum if and only if $E_d - M$ is a vertex-disjoint union of a minimum cycle Z passing through v, and possibly some zero cycles in $G_d = (V_d, E_d)$. (A similar result has appeared in [13].)

Proof: Let $M \subseteq E'/v$ be a complete matching of G'/v. As in the proof of Lemma 8, we can show that each vertex of G_d is adjacent to 0 or 2 edges in $E_d - M$, but vertex v is adjacent to exactly 2 edges in $E_d - M$. Thus $E_d - M$ is a vertex-disjoint union of cycles in G_d, one of these cycles passes through v. Conversely, let D be a vertex-disjoint union

of cycles in G_d, one of these cycles passes through v. One can easily verify that there exists a complete matching M of G'/v such that $D = E_d - M$. Clearly the weight of M is maximum if and only if the weight of D is minimum. Since D is minimum and G_d contains no negative cycle, D consists of a minimum cycle Z passing through v and possibly some zero cycles in G_d. Q.E.D.

Due to Lemma 10, one can find a minimum cycle passing through a specified vertex by finding a maximum weight matching, which takes time $O(n^{3/2} \log n)$. Thus a straightforward procedure for finding a minimum cycle in G_d would take $O(n^{5/2} \log n)$ time. (For each vertex v, find a minimum cycle passing through v.) In the following, we shall develop an $O(n^{3/2} \log n)$ algorithm for finding a minimum cycle.

Lemma 11. Let $v_1, v_2, \ldots, v_k$ be vertices of $G_d = (V_d, E_d)$. Then one can find k cycles $Z_1, Z_2, \ldots, Z_k$ in G_d in time $O(n^{3/2} \log n + kn \log n)$, where each Z_j is a minimum cycle passing through v_j, $j = 1, 2, \ldots, k$.

Proof: To find k minimum cycles $Z_1, Z_2, \ldots, Z_k$, we need to compute, for each vertex v_j, $j = 1, 2, \ldots, k$, a maximum weight matching of G'/v_j. We first find a maximum weight matching M of G', which takes $O(n^{3/2} \log n)$ time (Lemma 9). Then for each v_j, starting with M of G', we can find a maximum weight matching of G'/v_j in $O(n \log n)$ time [5] [2] since G'/v is constructed from G' by adding two vertices and two edges. Thus the total running time is $O(n^{3/2} \log n + kn \log n)$. Q.E.D.

Lemma 12. (Planar Graph Separator Theorem [11]) Let $G_d = (V_d, E_d)$ be a planar graph. Then V_d can be partitioned into three sets A, B and S

such that no edge joins a vertex in A with a vertex in B, $|A|$, $|B| \leq$ $c_1 |V_d|$, and $|S| \leq c_2 |V_d|^{1/2}$ where c_1 (< 1) and c_2 are two suitable positive constants.

Lemma 13. Let $G_d = (V_d, E_d)$ be a cubic planar graph which contains no negative cycle. Then a minimum cycle in G_d can be found in $O(n^{3/2} \log n)$ time.

Proof: We apply the planar graph separator theorem to G_d. Let A, B and S (separator) be the vertex partition asserted by Lemma 12, and let G_A and G_B be the subgraphs of G_d induced by A and B respectively. A minimum cycle in G_d is either a minimum cycle in G_A or G_B or a minimum cycle in G_d passing through a vertex in S. Thus a minimum cycle in G_d can be found by recursively finding a minimum cycle in G_A and a minimum cycle in G_B, and finding, for each vertex v in S, a minimum cycle in G_d passing through v. Let $T(n)$ be the running time of the algorithm on a graph having n vertices. Since S contains $O(n^{1/2})$ vertices, the minimum cycles passing through S can be computed in $O(n^{3/2} \log n)$ time due to Lemma 11. Then

$$T(n) = T(n_1) + T(n_2) + O(n^{3/2} \log n)$$

where $n_1 + n_2 \leq n$ and n_1, $n_2 \leq c_1 n$. An induction proof shows that $T(n) = O(n^{3/2} \log n)$.

Q.E.D.

Combining all the results we have obtained, we have the following major theorems.

Theorem 2. A maximum even-degree edge set of a cubic planar graph

$G_d = (V_d, E_d)$ can be found in $O(n^{3/2} \log n)$ time.

Theorem 3. A real-weight maximum cut of a planar graph $G = (V,E)$ can be found in $O(n^{3/2} \log n)$ time.

5. Concluding remarks

The contributions of this article are two-fold. First, the conventional maximum cut and minimum cut are unified to the more general real-weight maximum cut, and hence can be computed through a common framework. Second, a fast algorithm has been presented for finding a real-weight maximum cut of a planar graph. The algorithm makes extensive use of recent results on maximum matchings and minimum cycles, and achieves better performance than previous maximum cut-algorithms.

References:

1. K. Aoshima and M. Iri, Comments on F. Hadlock's paper: Finding a maximum cut of a planar graph in polynomial time, SIAM J. Comput., 6 (1977), pp. 86-87.

2. M. O. Ball and U. Derigs, An analysis of alternative strategies for implementing matching algorithms, Networks, 13 (1983), pp. 517-549.

3. S. Even, Graph Algorithms, Computer Science Press, Reading, Maryland, 1979.

4. H. N. Gabow, Implementation of algorithms for maximum matching on nonbipartite graphs, Ph.D. Thesis, Stanford University, Stanford, CA, 1974.

5. Z. Galil, S. Micali and H. Gabow, Priority queues with variable priority and an O(EV log V) algorithm for finding a maximal weighted matching in general graphs, 23rd Annual Symposium on Foundations of Computer Science, Chicago, 1982, pp. 255-261.

6. M. R. Garey and D. S. Johnson, Computers and Intractability, a Guide to the Theory of NP-completeness, W. H. Freeman and Company, San Francisco, 1979.

7. F. Hadlock, Finding a maximum cut of a planar graph in polynomial time, SIAM J. Comput, 4 (1975), pp.221-225.

8. T. C. Hu, Integer Programming and Network Flows, Addison-Wesley, Reading, Mass., 1969.

9. A. Itai and Y. Shiloach, Maximum flows in planar networks, SIAM J. Comput., 8 (1979), pp.135-150.

10. E. L. Lawler, Combinatorial Optimization: Networks and Matroids, Holt, Rinehart and Winston, New York, 1976.

11. R. J. Lipton and R. E. Tarjan, A separator theorem for planar graphs, SIAM J. Appl. Math., 36 (1979), pp. 177-189.

12. R. J. Lipton and R. E. Tarjan, Applications of a planar separator theorem, SIAM J. Comput, 9 (1980), pp. 615-627.

13. K. Matsumoto, T. Nishizeki and N. Saito, Planar multicommodity flows, maximum matchings and negative cycles, SIAM J. Comput., 15 (1986), pp. 495-510.

14. J. H. Reif, Minimum s-t cut of a planar undirected network in $O(n \log^2(n))$ time, SIAM J. Comput., 12 (1983), pp. 71-81.

15. Y. Shiloach, A multi-terminal minimum cut algorithm for planar graphs, SIAM J. Comput., 9 (1980), pp. 219-224.

16. W. T. Tutte, A short proof of the factor theorem for finite graphs, Canad. J. Math., 6 (1954), pp. 347-352.
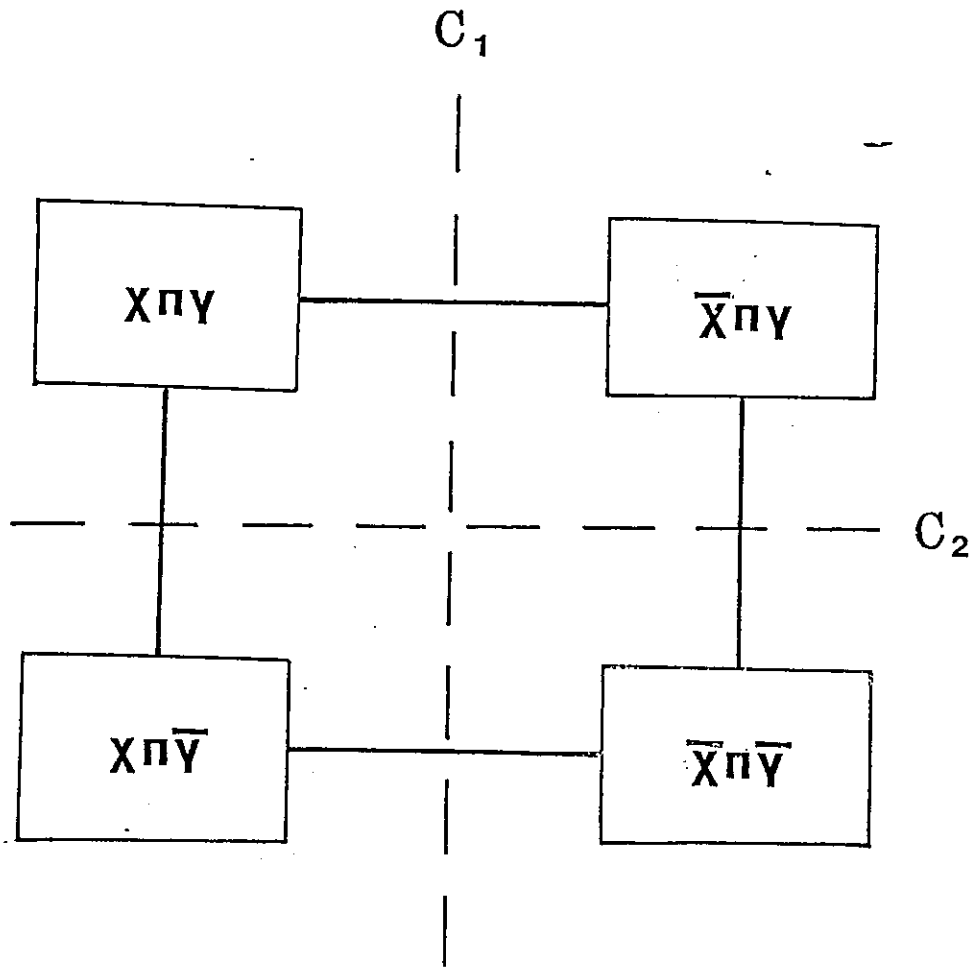
Fig. 1. Illustration for Lemma 1
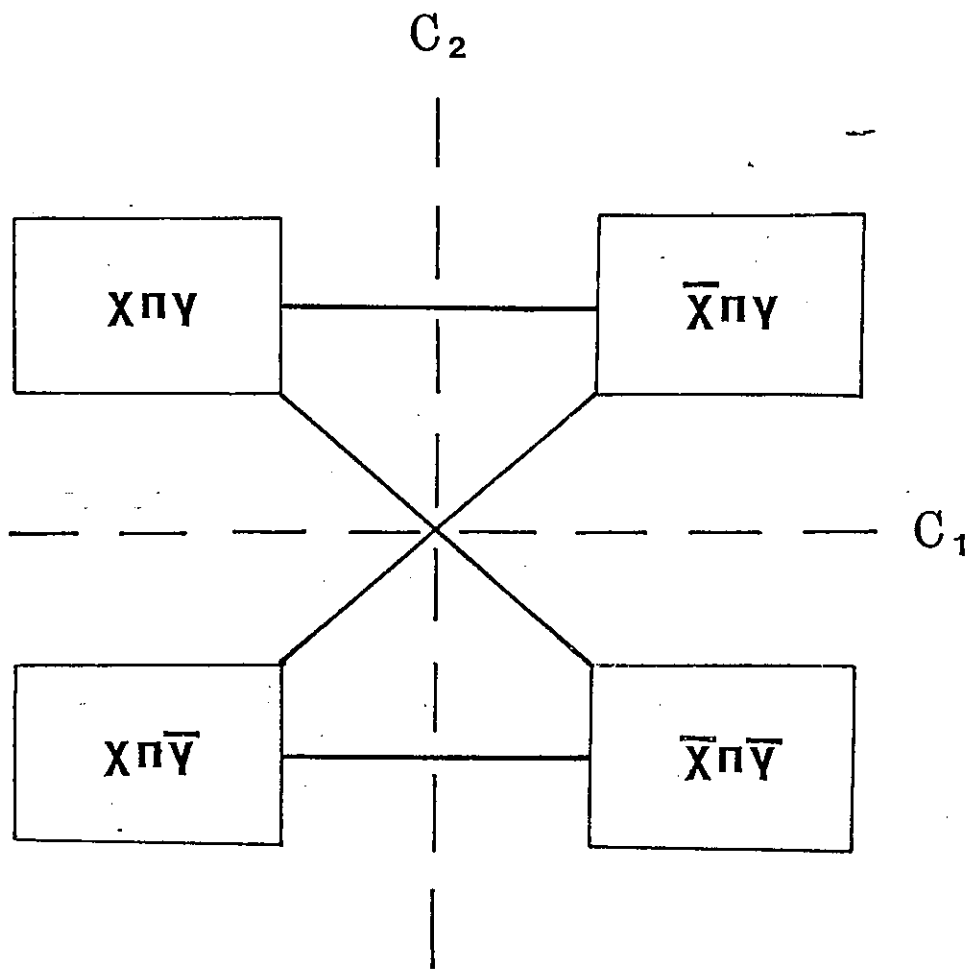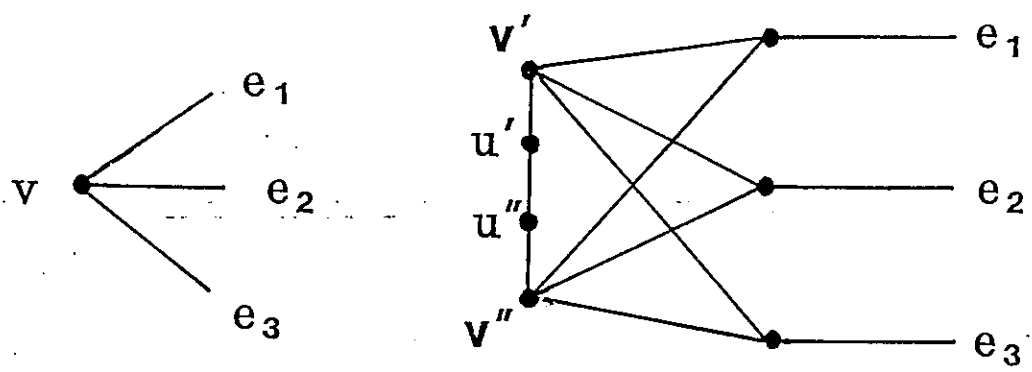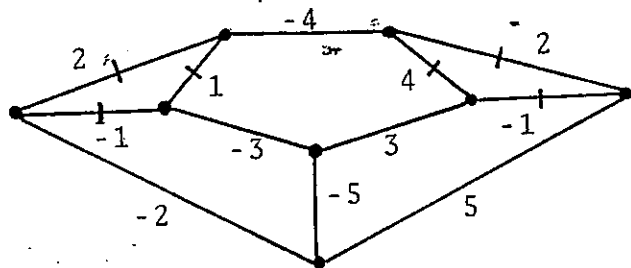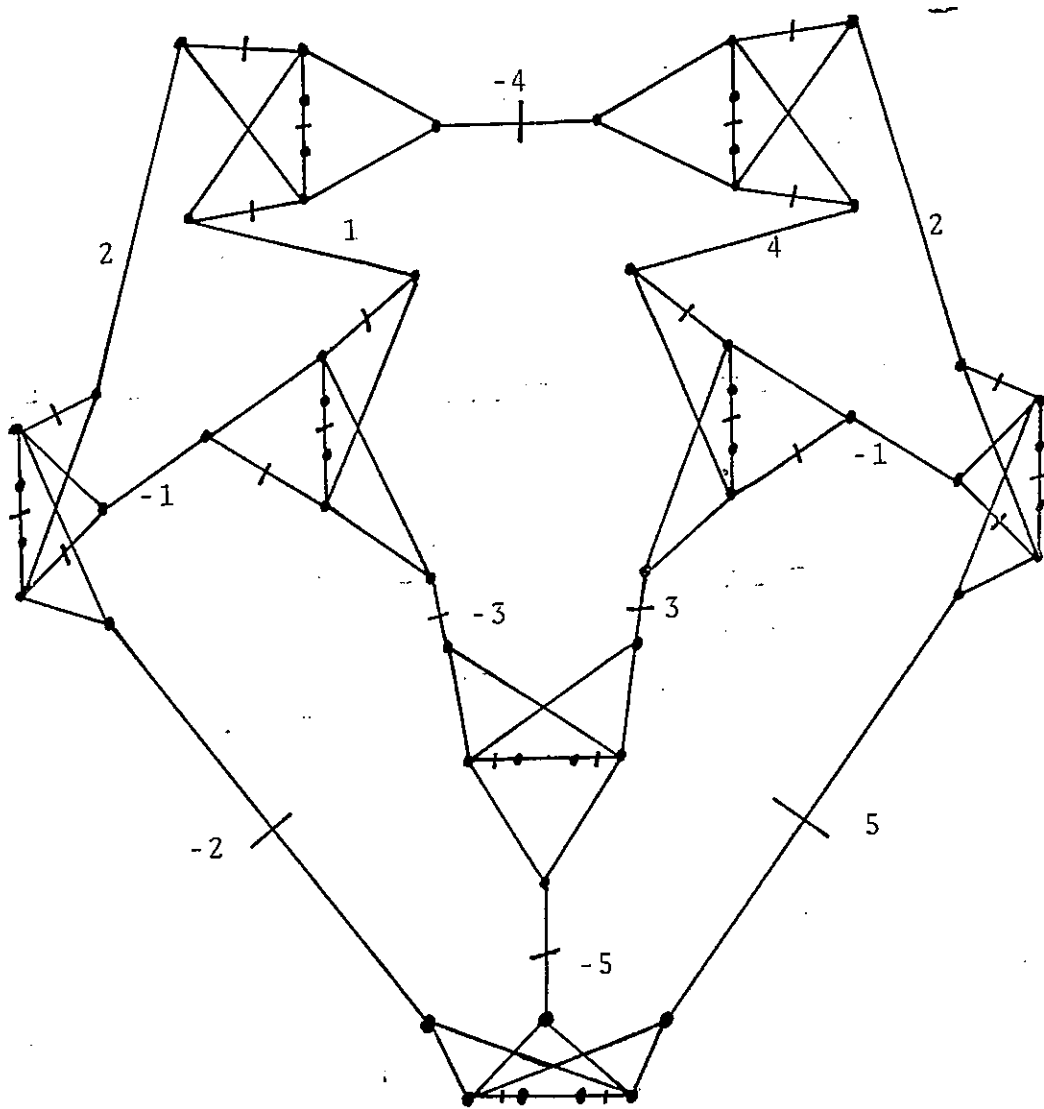
Fig. 2. Illustration for Lemma 2

Fig. 3. A star substituting vertex v

(a) G_d (Edges in D are marked by a short bar —┼—)
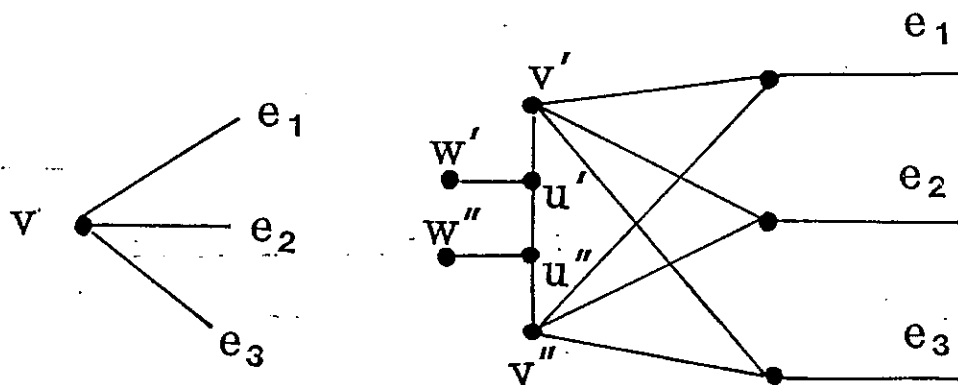

(b) G' (Edges in M are marked by a short bar —┼—)

Fig. 4. Graphs G_d and G'

Fig. 5. The star for vertex v in G'/v