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THE HOMOGRAPHY GROUP (PGL(2,F)) OF A MERIDIAN AS A LIBRA

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(by Kelly McKennon, November 2017)

Prologue

This paper is written in TEX `eplain`, and compiled using `pdf.tex`. This was done so that links and colors would be available without the restrictions of LaTex.

There are some symbols used here which are not universally standard. The most common are the following:

$$X \triangle Y$$

for the set complement of a subset Y of a set X ,

$$A \equiv B$$

for the statement: A is defined to be B ,

$$\phi(x)$$

for the value of a function ϕ at one of its arguments x ,

$$\llcorner\phi\lrcorner \quad \text{and} \quad \ulcorner\phi\urcorner\text{, respectively,}$$

for the domain and range, respectively, of a function ϕ ,

$$\overrightarrow{\phi}(S) \equiv \{\phi(x) : x \in S\}$$

for the image of a subset S of the domain of a function ϕ ,

$$\phi | X \hookrightarrow Y$$

for a function from a set X into a set Y ,

$$\mathbb{N}$$

for the set of natural numbers 1, 2, 3. . . ,

$$\overleftrightarrow{a,b}$$

for the line determined by two distinct points a and b of a projective space,

$$\underline{n} \equiv \{1,2,\ldots n\}$$

for the set of the first n positive integers,

$$X^{Y}$$

for the family of all functions from a set Y to a set X and

$$Y^{X}!$$

for the family of all bijections from X onto Y .

We also use a few non-standard terms. As a “singleton” refers of a set with exactly one element, a **doubleton** shall be a set with exactly two elements, a **tripleton** shall be a set with exactly three elements and a **quadrupleton** shall be a set with exactly four elements.

Some of the material in these papers appeared originally in http://vixra.org/abs/1306.0233 , and several mistakes in that paper have been rectified here.

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Table of Contents

1. Introduction

(1.1) Purpose In the monograph The Projective Line as a Meridian we dealt with the structure of a meridian, or one-dimensional projective space. What makes one meridian different from another is its group of homographies. Here our attention is focused on this family of homographies and its structure. In the case of the circle meridian – the meridian of which the associated field is the field of real numbers – this structure is closely related to the geometry of a circular hyperbola in euclidan 3-space. Furthermore, this example provides considerable motivation for the general theory.

Before we commence however, we shall review a part of what was covered in the previous mongograph about meridians.

(1.2) Definitions of a Meridian A **meridian** M is a set of cardinality at least 4, with a group $\mathcal{G}$ of permutations, called **homographies**, such that the following three conditons are satisfied:

$$(\forall\,\{a,b,c\}\subset M \text{ and } \{x,y,z\}\subset M \text{ tripletons})(\exists!\,\phi\in\mathcal{G})\quad \phi(a)=x\,,\ \ \phi(b)=y \quad\text{and}\quad \phi(c)=z\,, \tag{1}$$

$$(\forall\,\{a,b,c\}\subset M \text{ a tripleton})(\exists!\,\phi\in\mathcal{G})\quad \phi(a)=b\,,\ \ \phi(b)=c \quad\text{and}\quad \phi\circ\phi\circ\phi\circ\phi=\iota_M \tag{2}$$

and

$$(\forall\,\phi\in\mathcal{G}\colon\ (\exists\,\{a,b\}\subset M \text{ a doubleton})\ \phi(a)=b \text{ and } \phi(b)=a)\quad \phi\circ\phi=\iota_M \tag{3}$$

(where ι_M is the identity function on M). The set $\mathcal{G}$ is said to be a **meridian group of permutations of the set** M.

A **meridian family $\mathcal{M}$ of involutions** on a set M is a family of self-inverse permutations of M for which the following three conditions hold:[1]

$$(\forall\,\{a,b\}\subset M \text{ a doubleton})(\forall\,\{c,d\}\subset M\,\triangle\,\{a,b\} \text{ a doubleton})(\exists!\,\phi\in\mathcal{M})\quad \phi(a)=c \text{ and } \phi(b)=d\,, \tag{4}$$

$$(\forall\,\{a,b\}\subset M)\quad \{\phi\in\mathcal{M}:\phi(a)=b\} \text{ is a libra} \tag{5}$$

and

$$(\forall\,\{\alpha,\beta\}\subset\mathcal{M})\quad \alpha\circ\beta\circ\alpha\in\mathcal{M}\,. \tag{6}$$

If M is a meridian with group of homographies $\mathcal{G}$, then the family of self-inverse permutations in $\mathcal{G}$ is a meridian family of involutions. Conversely, if $\mathcal{M}$ is a meridian family of involutions on a set M, then the smallest group of permutations containing $\mathcal{M}$ as a subset, is a group of homographies relative to which M is a meridian.

(1.3) Homographies We shall say that $\phi\in\mathcal{G}$ is **pure rotation** if it has no fixed points, a **translation** if it has exactly one fixed point and a **dilation** if it has two fixed points. In some meridians there are no pure rotations, but there are always translations and dilations.

An **involution** ϕ equals its own inverse. An involution can never be a translation: it is either a dilation or a pure rotation. Each homography which is not an involution is the composition of two distinct involutions. What is more, for any $\theta\in\mathcal{G}$, we have

$$\theta \text{ is an involution} \iff (\exists\,\{a,b\}\subset M \text{ a doubleton})\quad \theta(a)=b \quad\text{and}\quad \theta(b)=a\,, \tag{1}$$

$$\theta \text{ is a translation} \iff (\exists\,\{\pi,\sigma\}\subset\mathcal{M} \text{ dilations with a single common fixed point})\quad \theta=\pi\circ\sigma\,, \tag{2}$$

[1] Being a libra in condition (5) means that $(\forall\,\{\alpha,\beta,\alpha\}\in\mathcal{M})\quad \alpha\circ\beta\circ\gamma$ is in $\mathcal{M}$.

$$\theta \text{ is a dilation } \Longleftrightarrow \theta \text{ is not a translation and either } \theta \text{ is an involution or } (\exists\, \pi \in \mathcal{M} \text{ a dilation and } \sigma \in \mathcal{M} \text{ which agree on two points}) \quad \theta = \pi \circ \sigma\,, \tag{3}$$

$$\theta \text{ is a pure rotation } \Longleftrightarrow \text{ either } \theta \text{ is an involution with no fixed point or } (\exists\, \pi \in \mathcal{M} \text{ a dilation and } \sigma \in \mathcal{M} \text{ which agree on no point}) \quad \theta = \pi \circ \sigma \tag{4}$$

and
$$\mathcal{G} \;=\; \{\pi \circ \sigma : \{\pi, \sigma\} \subset \mathcal{M} \text{ and } \pi \text{ is a dilation}\}\,. \tag{5}$$

We shall write the function ϕ of (1.2.1) as

$$\boxed{\begin{smallmatrix} a & b & c \\ x & y & z \end{smallmatrix}}\,. \tag{6}$$

(1.4) Harmony Let M be a meridian with families $\mathcal{G}$ of homographies and $\mathcal{M}$ of involutions. Two disjoint doubletons $\{a,b\} \subset M$ and $\{c,d\} \subset M$ are said to be **harmonic pairs** if there exists a function $\phi \in \mathcal{G}$ such that the equalities in (1.2.2) hold and that $\phi(d) = a$. This is equivalent to each of the following conditions:

$$(\exists\, \phi \in \mathcal{M}) \quad \phi(a) = b,\ \phi(c) = c \text{ and } \phi(d) = d \tag{1}$$

and

$$(\exists\, \phi \in \mathcal{G} \text{ a translation}) \quad \phi(a) = b\,, \phi(b) = c \text{ and } \phi(d) = d\,. \tag{2}$$

For any tripleton $\{a,b,c\}$, there is exactly one point $d \in M$ such that the sets $\{a,b\}$ and $\{c,d\}$ are harmonic pairs. The symmetry induced by this fourth point d is at the heart of a considerable part of mathematics, and so we shall introduce notation to express it:

$$a \overset{c}{\ddagger} b \equiv d\,. \tag{3}$$

(1.5) Meridian Quinary Operator If $\{a,b\}$ and $\{c,d\}$ are as in (1.2.4), and if e is any element of L, there is a unique element $\boxed{\begin{smallmatrix} & c & \\ a & e & b \\ & d & \end{smallmatrix}}$ of M such that the element of $\mathcal{M}$ which takes a to b and c to d, also takes e to $\boxed{\begin{smallmatrix} & c & \\ a & e & b \\ & d & \end{smallmatrix}}$. This defines the **meridian quinary operator on** M.

We note that, if $\{a,b,c\} \subset M$ is a tripleton, then the unique fourth point d such that $\{a,b\}$ and $\{c,d\}$ are harmonic pairs is just

$$a \overset{c}{\ddagger} b \;=\; \boxed{\begin{smallmatrix} & b & \\ a & c & a \\ & b & \end{smallmatrix}}\,. \tag{1}$$

(1.6) Meridian Fields For any tripleton $\{0, 1, \infty\} \subset M$, the meridian quinary operator may be used to construct a field $F \equiv M \triangle \{\infty\}$: we define

$$(\forall\,\{x,y\}\subset F)\quad x+y\equiv\boxed{\begin{matrix} & x & \\ \infty & 0 & \infty \\ & y & \end{matrix}}\quad\text{and}\quad x\cdot y\equiv\boxed{\begin{matrix} & x & \\ \infty & 1 & 0 \\ & y & \end{matrix}}\quad(\text{if }\{0,\infty\}\neq\{x,y\})\,. \tag{1}$$

Such a field F will be called a **field of the meridian** M. Any two fields of M are isomorphic as fields.

We recall that the **characteristic of a field** is the least positive integer n such that $n\cdot x=0$ for some $x\neq 0$, if there is such an n. If there is not such an n, the field is said to be of characteristic zero. We define the **characteristic of a meridian** to be the characteristic of any one of its fields.

Having determined such a field, one can describe the homographies of M in terms of quadruples $\{a,b,c,d\}\subset M$ such that $a\cdot d\neq b\cdot c$. For such a quadruple we define

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}(x)\equiv\begin{cases}\dfrac{a\cdot x+b}{c\cdot x+d} & \text{if } x\in F \text{ and } c\cdot x+d\neq 0;\\ \dfrac{b}{d} & \text{if } x=\infty \text{ and } d\neq 0;\\ \infty & \text{otherwise.}\end{cases} \tag{2}$$

Is convenient to use this "matrix" notation since the matrix of compositions of homographies corresponds to the matrix product of the corresponding matrices.[2]

If any meridian field is isomorphic to the field of real numbers, we say that M is a **circular meridian**.[3] If any meridian field is isomorphic to the field of complex numbers, we say that the meridian is a **spherical meridian**.

(1.7) Meridian Isomorphisms and Automorphisms If M is a meridian with homography group $\mathcal{G}$ and M′ another meridian with homography group $\mathcal{G}'$, a bijection $\phi|M\hookrightarrow M'$ is said to be a **meridian isomorphism** if

$$\mathcal{G}=\{\phi^{-1}\circ\theta\circ\phi:\theta\in\mathcal{G}'\}\,. \tag{1}$$

Condition (1) is equivalent to

$$\mathcal{M}=\{\phi^{-1}\circ\theta\circ\phi:\theta\in\mathcal{M}'\} \tag{2}$$

where $\mathcal{M}$ and $\mathcal{M}'$, respectively, are the meridian families of involutions on M and M′, respectively.

If M and M′ are identical, a meridian isomorphism is called a **meridian automorphism**. Every homography of a meridian is an automorphism, but the converse is not always true, as in the case of spherical meridians, where complex conjugation extends to an automorphism of M which is not a homography.

(1.8) Wurf Meridians and Cross Ratios Let M be a meridian, and let

$$\mathcal{W}\equiv\{[a,b,c,d]\in M\times M\times M\times M:\#\{a,b,c,d\}\geq 3\}\,.$$

Let $\mathfrak{W}$ be the family of equivalence classes of M where the equivalence relation $\sim$ is given by

$$(\forall\,\{a,b,c,d\}\in\mathcal{W}\text{ and }\{w,x,y,z\}\in\mathcal{W}) \tag{1}$$

---

[2] Two such "matrices" define the same homography if, and only if, one is a non-0 multiple of the other.

[3] One reason that the term "circular" is here associated with the real field is that, in this case, the set M may be associated in a topological way with the circle. Furthermore, regarding M as a circle gives insight into $\mathcal{M}$, as is illustrated in (1.9) *infra*.

$$\{a,b,c,d\}\sim\{w,x,y,z\} \iff (\exists\, \phi\in\mathcal{G})\ \ [\phi(a),\phi(b),\phi(c),\phi(d)]=[w,x,y,z]\,.$$

We call the elements of $\mathfrak{W}$ **wurfs**.[4]

Let $\{\infty,0,1\}$ be a tripleton in M. Then the ordered triple $[\infty,0,1]$ induces a function

$$\kappa\Big|\mathcal{W}\ni[a,b,c,d]\hookrightarrow \boxed{\begin{smallmatrix} a & b & c\\ \infty & 0 & 1\end{smallmatrix}}(d)\in M\,. \tag{2}$$

Such a function is called a **cross ratio**. It can be computed using the field operations of the field determined by ∞, 0 and 1. Its value for each $[a,b,c,d]\in\mathcal{W}$ is given by

$$\kappa([a,b,c,d])=\frac{(a\text{-}c)\cdot(b\text{-}d)}{(b\text{-}c)\cdot(a\text{-}d)}\ . \tag{3}$$

One virtue of cross ratios κ is that they may be used to test whether a bijective function $\phi\big|M\hookrightarrow M$ is a homography or not:

$$\phi\in\mathcal{G} \iff (\forall\,[a,b,c,d]\in\mathcal{W})\ \ \kappa([\phi(a),\phi(b),\phi(c),\phi(d)])=\kappa([a,b,c,d])\,. \tag{4}$$

Cross ratios are constant on each wurf and so induce bijections from $\mathfrak{W}$ onto M: we shall write $\mathcal{L}$ for the family of all such. For each $\phi\in\mathcal{L}$, the set

$$\{\phi^{-1}\circ\theta\circ\phi:\theta\in\mathcal{G}\} \tag{5}$$

is independent on the choice of ϕ. Then family of (5) satisfies axioms (1.2.1) through (1.2.3) and so confers on $\mathfrak{W}$ eligibility for the title of meridian. Each of the functions in $\mathcal{L}$ is then a meridian isomorphism.

If a field of the meridian M does not have characteristic 3, then the meridian $\mathfrak{W}$ has six special elements:

$$\mathfrak{a}=\{[a,b,c,d]:a=d \text{ or } b=c\}\,, \tag{6}$$

$$\mathfrak{b}=\{[a,b,c,d]:b=d \text{ or } a=c\}\,, \tag{7}$$

$$\mathfrak{c}=\{[a,b,c,d]:c=d \text{ or } a=b\}\,, \tag{8}$$

$$\mathfrak{a}'=\{[a,b,c,d]:\{a,d\} \text{ and } \{c,d\} \text{ are harmonic pairs}\}\,, \tag{9}$$

$$\mathfrak{b}'=\{[a,b,c,d]:\{b,d\} \text{ and } \{a,c\} \text{ are harmonic pairs}\} \tag{10}$$

and

$$\mathfrak{c}'=\{[a,b,c,d]:\{c,d\} \text{ and } \{a,b\} \text{ are harmonic pairs}\}\,. \tag{11}$$

Relative to the quinary operator on $\mathfrak{W}$, these six are related as follows:

$$\boxed{\begin{smallmatrix} & \mathfrak{c} & \\ \mathfrak{a} & \mathfrak{c}' & \mathfrak{b}\\ & \mathfrak{c} & \end{smallmatrix}}=\mathfrak{c}'\,,\quad \boxed{\begin{smallmatrix} & \mathfrak{b} & \\ \mathfrak{a} & \mathfrak{b}' & \mathfrak{c}\\ & \mathfrak{b} & \end{smallmatrix}}=\mathfrak{b}' \quad\text{and}\quad \boxed{\begin{smallmatrix} & \mathfrak{a} & \\ \mathfrak{b} & \mathfrak{a}' & \mathfrak{c}\\ & \mathfrak{a} & \end{smallmatrix}}=\mathfrak{a}'$$

If the characteristic of M is 3, then $\{\mathfrak{a},\mathfrak{b},\mathfrak{c}\}$ is still a tripleton, but $\mathfrak{a}'=\mathfrak{b}'=\mathfrak{c}'$.

(1.9) Example: Circle Meridian Let **P** be 2-dimensional euclidean space. For each line L in **P** we shall add a point $\infty(L)$ to L not in **P**. We do this in such a way that points $\infty(L_1)$ and $\infty(L_2)$ are equal if, and only if, the lines L_1 and L_2 are parallel. The set $\infty(\mathbf{P})$ of all these "points at infinity" is called the **line at infinity** and its union with **P** will be denoted by

[4] The term "wurf", or more properly "Wurf", was introduced by Karl von Staudt in his mid-eighteenth century work providing a synthetic foundation of projective geometry.

$$\mathbb{P}\,. \tag{1}$$

By a **line in** $\mathbb{P}$, we shall mean either $\infty(\mathbf{P})$ or $L \cup \{\infty(L)\}$, where L is a line in **P**.

Let C be a circle in **P**. Let x and y be elements of C. By $\overleftrightarrow{x,y}$ we shall mean the line through x and y. If p is any point in $\mathbb{P} \triangle C$, and x is any point in C, then the line $\overleftrightarrow{p,x}$, unless it is tangent to C, intersects C at exactly one other point: we write

$$p_C(x) \tag{2}$$

for this other point. In the case of tangency, we define the value of (2) to be just x. It is evident that each such function p_C thus defined is an involution on C. In fact, it can be shown that the family

$$\mathcal{M}(C) \equiv \{p_C : p \in \mathbb{P} \triangle C\} \tag{3}$$

is a meridian family of involutions. Relative to this family, C is said to be a **circle meridian**.

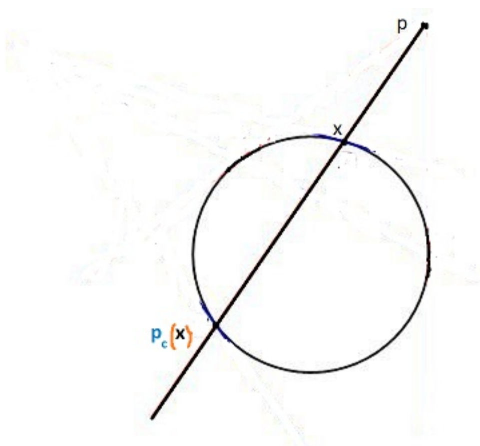


Fig. 1: Circle Meridian Involution

Each circle meridian is isomorphic to every other circle meridian as a meridian.[5]

In the following figure is illustrated a harmonic pair of pairs {a,c} and {b,d} in the circle meridian:

[5] The same constructions which have here been adopted for a circle can be adduced for an ellipse, or even for a hyperbola or parabola in **P**, if one includes the relevant points in $\infty(\mathbf{P})$. Each meridian obtained in this manner is isomorphic to a circle meridian.

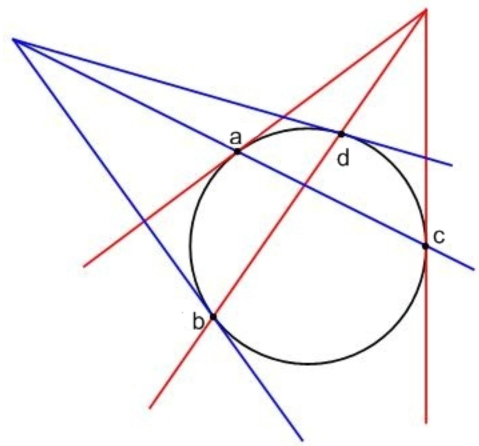


Fig. 2: Harmonic Pairs on the Circle

In the next figure the quinary operator is illustrated for the circle meridian:

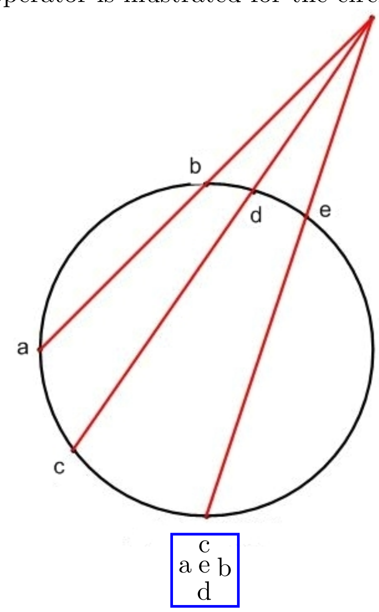


Fig. 3: Quinary Operator on the Circle Meridian

The meridian automorphisms of a circle are just the homographies of that circle.

(1.10) Example: Line Meridian Let L be any line in $\mathbb{P}$ such that $L \cap \mathbf{P} \neq \emptyset$. Let C be a circle in **P** tangent to L at some point p, and let o be any point on C distinct from p. For each point x of C let $\omega(x)$ be the intersection point of the two lines L and $\overleftrightarrow{o,x}$, where $\overleftrightarrow{o,o}$ is defined to be the line tangent to C at o.[6] The mapping $\omega \big| C \hookrightarrow L$ thus created is a bijection. The the family

$$\mathcal{L} \equiv \{\omega \circ \phi \circ \omega^{-1} : \phi \in \mathcal{M}(C)\} \tag{1}$$

is a meridian family of involutions for L. Although this definition seems to be dependent on the choice of

[6] This mapping is the 2-dimensional avatar of the so-called stereographic projection of a sphere onto a tangent plane.

C , it actually is not, and we can describe $\mathcal{L}$ directly in terms of the geometry of $\mathbb{P}$.

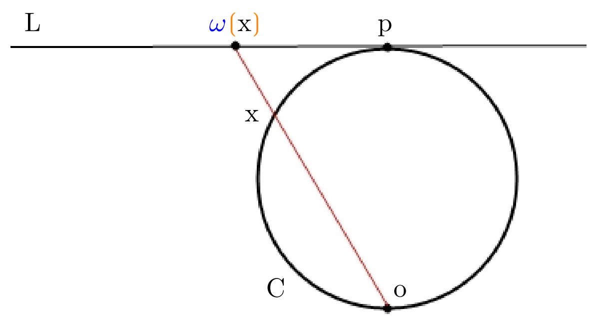


Fig. 4: Meridian Isomorphism from a Circle onto a Line

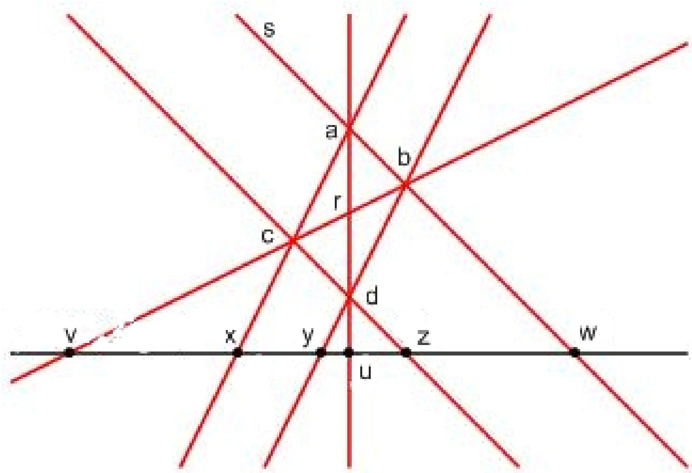


Fig. 5: Quinary Operator on the Line Meridian

If $\{a,b,c,d\}\subset\mathbb{P}$ is a quadrupleton such that no three of its elements are collinear, we say that it generates a **complete quadrilateral**, the **sides** of which are $\overleftrightarrow{a,b}$, $\overleftrightarrow{a,c}$, $\overleftrightarrow{a,d}$, $\overleftrightarrow{b,c}$, $\overleftrightarrow{b,d}$ and $\overleftrightarrow{c,d}$. It can be shown that for such a complete quadrilateral

$$(\overleftrightarrow{c,d}\wedge L) \ = \ \boxed{\begin{array}{ccc} & (\overleftrightarrow{a,c}\wedge L) & \\ (\overleftrightarrow{a,d}\wedge L) & (\overleftrightarrow{a,b}\wedge L) & (\overleftrightarrow{d,c}\wedge L) \\ & (\overleftrightarrow{b,d}\wedge L) & \end{array}} \ , \tag{2}$$

where, for two distinct lines U and V , $U\wedge V$ is defined to be the point of which the singleton is $U \triangle V$. Thus, to compute $z\equiv\boxed{\begin{array}{ccc} & w & \\ u & y & v \\ & x & \end{array}}$ for $\{u,v,w,x,y\}\subset L$ with u, v and w distinct, we may proceed as follows: Take $r\in\mathbb{P}$ to be any point not on L . Let $s\in\mathbb{P}$ be any point of $\mathbb{P}\triangle L$ such that $r\notin\overleftrightarrow{w,s}$. Let

$$a \equiv (\overleftrightarrow{u,r} \wedge \overleftrightarrow{w,s}),\ b \equiv (\overleftrightarrow{v,r} \wedge \overleftrightarrow{w,s}),\ c \equiv (\overleftrightarrow{x,a} \wedge \overleftrightarrow{v,r}) \text{ and } d \equiv (\overleftrightarrow{y,b} \wedge \overleftrightarrow{u,r}). \tag{3}$$

Then (2) implies that

$$z = \boxed{\begin{smallmatrix} & w & \\ u & y & v \\ & x & \end{smallmatrix}} = (\overleftrightarrow{c,d} \wedge L). \tag{4}$$

(1.11) Example: Sphere Meridian Let E denote three dimensional Euclidian space. As in P we associate to each line L a point $\infty(L)$ not in E in such a way that, for two lines L_1 and L_2, the points $\infty(L_1)$ and $\infty(L_2)$ are equal if, and only if, the two lines are parallel. The set

$$\infty(\mathsf{E}) \equiv \{\infty(L) : L \text{ is a line in } \mathsf{E}\} \tag{1}$$

is called the **plane at infinity**. We denote

$$\mathbb{E} \equiv \mathsf{E} \cup \infty(\mathsf{E}) \tag{2}$$

and call $\mathbb{E}$ a **three dimensional real projective space**. A subset X of $\infty(\mathsf{E})$ is a **line at infinity** if

$$(\exists\, P \text{ a plane in } \mathsf{E}) \quad X = \{\infty(L) : L \text{ is a line in } P\}. \tag{3}$$

By a **line in** $\mathbb{E}$, we shall mean either a line at infinity or $L \cup \{\infty(L)\}$, where L is a line in E. By a **plane in** $\mathbb{E}$, we shall mean either the plane at infinity or a set of the form $N \cup \{\infty(L) : L \text{ is a line in } N\}$, where N is a plane in E.

Let now S be a sphere in E. Let L be any line in $\mathbb{E}$ not tangent to S, which intersects S. Then there are two intersection points p and q. If P is the tangent plane at p and Q is the tangent plane at q, then $P \cap Q$ is called the **line dual to L relative to the sphere** S. Given any two lines L and M in $\mathbb{E}$ which do not intersect, and any point $x \notin (L \cup M)$, there exists exactly one line

$$\overleftrightarrow{(L,M;x)} \tag{4}$$

which passes through x and intersects both L and M. If L is as above, and x is any point on S, the line $\overleftrightarrow{(L,P \cap Q;x)}$ intersects S at another point

$$\boxed{S;L}(x) \tag{5}$$

(where $\boxed{S;L}(x)$ is defined to be x if x is on L). The family

$$\mathcal{M}(S) \equiv \{\boxed{S;L} : L \text{ a line in } \mathbb{P} \text{ with } \#(L \cap S) = 2\} \tag{6}$$

is a meridian family of involutions of S. The meridian S, relative to this family, will be called a **sphere meridian**.[7]

A plane which intersects S at more than one point, intersects S in a circle. The harmonic pairs of S are just the pairs which are on such a circle and harmonic relative to that circle.

[7] When an ordered basis $[0,\infty,1]$ for S is chosen with 0 and ∞ at opposite ends of a diameter of S and 1 equidistant from 0 and ∞, S is sometimes called a **Riemann Sphere**.

Given any point p of $\mathbf{E}$ not on S, the involution of S obtained by taking lines through p which intersect S is not a homography of S, but it is a meridian automorphism.

2. The Idea of a Libra

(2.1) Purpose In The Projective Line as a Meridian we introduced the idea of a libra, laying out its first properties and promising later to explain how its name derives from a set of scales.

Fig. 6: Scales

In this section we shall fulfill this promise.

(2.2) Scales Suppose that we have a pair of scales suspended from the ends of a beam. Suppose that on each scale there are three equidistant places on which we can place weights taken from a set L: we shall identify these places on one scale by $\spadesuit$, $\clubsuit$ and $\bigstar$; and on the other scale by $\heartsuit$, $\diamondsuit$ and $\maltese$. Suppose that if we picked up one of the scales, and without turning it, set it down on the other — then the order of the places in clockwise direction would be

$$\spadesuit, \heartsuit, \clubsuit, \diamondsuit, \bigstar \text{ and } \maltese . \tag{1}$$

We shall write Σ for the set $\{\spadesuit, \heartsuit, \clubsuit, \diamondsuit, \bigstar, \maltese\}$. Thus, a clockwise rotation of both scales by an angle of $2\pi/3$ would be equivalent to applying the permutation

$$\sigma \equiv \{[\spadesuit,\clubsuit],[\heartsuit,\diamondsuit],[\clubsuit,\bigstar],[\diamondsuit,\maltese],[\bigstar,\spadesuit],[\maltese,\heartsuit]\} . \tag{2}$$

to the set Σ.

In placing a **load** on the scales we are defining a function from Σ to L. Certain loads will leave the scales in equilibrium: we shall write $\mathcal{E}$ for the family of such loads. Such a subfamily of L^{Σ} will be called an **equilibrium family for** L provided that the following three postulates are fulfilled:

$$(\forall\,\theta\in \mathrm{L}^{\Sigma})\quad \theta\in\mathcal{E} \iff \theta\circ\sigma\in\mathcal{E}\,, \tag{3}$$

$$(\forall\,\theta\big|\{\spadesuit,\heartsuit,\clubsuit,\diamondsuit,\bigstar\}\hookrightarrow \mathrm{L})(\exists!\,x\in \mathrm{L})\ \theta\cup\{[\maltese,x]\}\in\mathcal{E} \tag{4}$$

and
$$(\forall\,\theta\big|\{\spadesuit,\heartsuit,\clubsuit,\diamondsuit\}\hookrightarrow \mathrm{L})(\forall\,\{x,y\}\subset \mathrm{L})\quad \theta\cup\{[\bigstar,x],[\maltese,x]\}\in\mathcal{E} \iff \theta\cup\{[\bigstar,y],[\maltese,y]\}\in\mathcal{E}\,. \tag{5}$$

The effect of (3) is that rotating both scales by the same angle does not affect whether a load is in equilibrium. The effect of (4) is that, if five of the places on the scales are loaded, then there is exactly one choice for the sixth place which will puts the scales into equilibrium. The effect of (5) is that if a load is in equilibrium and two adjacent (in the sense of (1)) places have the same element of L , then the load will still be in equilibrium if both these places are given a different equal value.

(2.3) Associated Trinary Libra Operator Let $\mathcal{E}$ be a family of equilibria for a set L . **The associated libra trinary operator**

$$\lfloor,,\rfloor\big|\mathrm{L}\times\mathrm{L}\times\mathrm{L}\ni[x,y,z]\hookrightarrow\lfloor x,y,z\rfloor\in\mathrm{L}$$

is defined by

$$(\forall\,\{x,y,z\}\subset \mathrm{L})\quad \{[\spadesuit,x],[\heartsuit,y],[\clubsuit,z],[\diamondsuit,z],[\bigstar,z],[\maltese,\lfloor x,y,z\rfloor]\}\in\mathcal{E}\,. \tag{1}$$

The postulates (2.2.3), (2.2.4) and (2.2.5) imply each of the following properties:

$$(\forall\,\{x,y\}\subset \mathrm{L})\quad \lfloor x,y,y\rfloor = x = \lfloor y,y,x\rfloor \tag{2}$$

$$(\forall\,\{a,b,c,d,e\}\subset \mathrm{L})\quad \lfloor\lfloor a,b,c\rfloor,d,e\rfloor = \lfloor a,b,\lfloor c,d,e\rfloor\rfloor\,. \tag{3}$$

and
$$(\forall\,\{a,b,c,d,e\}\subset \mathrm{L})\quad \lfloor a,\lfloor d,c,b\rfloor,e\rfloor = \lfloor a,b,\lfloor c,d,e\rfloor\rfloor\,. \tag{4}$$

(2.4) Trinary Libra Operator Let L be any set and $\lfloor,,\rfloor\big|\mathrm{L}\times\mathrm{L}\times\mathrm{L}\ni[x,y,z]\hookrightarrow\lfloor x,y,z\rfloor\in\mathrm{L}$ any trinary operator on L for which

$$(\forall\,\{x,y\}\subset \mathrm{L})\quad \lfloor x,y,y\rfloor = x = \lfloor y,y,x\rfloor \tag{1}$$

and

$$(\forall\,\{a,b,c,d,e\}\subset \mathrm{L})\quad \lfloor\lfloor a,b,c\rfloor,d,e\rfloor = \lfloor a,b,\lfloor c,d,e\rfloor\rfloor \tag{2}$$

hold. Then $\lfloor,,\rfloor$ is said to be a **libra trinary operator on** L .
For such an operator we may define the family

$$\mathcal{E}\equiv\ \{\theta\in\mathrm{L}^{\Sigma}:\lfloor\lfloor\theta(\spadesuit),\theta(\heartsuit),\theta(\clubsuit)\rfloor,\theta(\diamondsuit),\ \theta(\bigstar)\rfloor=\theta(\maltese)\}\,. \tag{3}$$

One can show that such a family is a family of equilibrium for the set L and that the associated trinary libra operator is just $\lfloor,,\rfloor$. We shall say that L is a **libra relative to $\mathcal{E}$** and **relative to $\lfloor,,\rfloor$** .

The various compositions of libra operators with libra operators, in view of (2.3.1), (2.3.2) and (2.3.3), may be greatly simplified: we define

$$\lfloor a,b,c,d,e\rfloor\ \equiv\ \lfloor\lfloor a,b,c\rfloor,d,e\rfloor\ =\ \lfloor a,\lfloor d,c,b\rfloor,e\rfloor\ =\ \lfloor a,b,\lfloor c,d,e\rfloor\rfloor\,. \tag{4}$$

Each composition of libra operators may be converted to a form

$$\lfloor a_1,a_2,\lfloor a_3,a_4,\lfloor \ldots \lfloor a_{n-2},a_{n-1},a_n \rfloor \ldots \rfloor\rfloor\rfloor \tag{5}$$

for n an odd positive integer. We shall at times adopt the abbreviation

$$\lfloor a_1,a_2,\ldots,a_n \rfloor \tag{6}$$

for (5).

(2.5) Libra Groups Let L be a libra relative to a libra trinary operator $\lfloor,,\rfloor$, and let e be any element of L . We define

$$(\forall\, \{x,y\} \subset L) \quad x{\cdot}y \equiv \lfloor x,e,y \rfloor\,. \tag{1}$$

It is not difficult to show that $\cdot$ is a group binary operation for which e is the identity element and

$$(\forall\, x \in L) \quad x^{-1} = \lfloor e,x,e \rfloor\,. \tag{2}$$

We shall say that $\cdot$ is the **libra group operator determined by the element** e **of** L .

Now let $\cdot$ be any binary group operation on a set G and define

$$(\forall\, \{x,y,z\} \subset G) \quad \lfloor x,y,z \rfloor \equiv x{\cdot}y^{-1}{\cdot}z\,. \tag{3}$$

Then $\lfloor,,\rfloor$ is a trinary libra operator for G . Furthermore, if e denotes the identity of G , then $\cdot$ is the libra group operator determined by e .

Thus there is a close connection between libra operators and group operators. For this reason we say that a libra family $\mathcal{E}$ of equilibrium is **abelian** if each permutation of Σ which sends red elements to red elements and black elements to black elements, preserves equilibrium loads. We say that a libra trinary operator is **abelian** provided that

$$(\forall\, \{x,y,z\} \subset G) \quad \lfloor x,y,z \rfloor = \lfloor z,y,x \rfloor\,. \tag{4}$$

It is not difficult to show that a libra family of equilibrium is abelian, if and only if its associated libra operator is abelian, if and only if the associated group operators are all abelian, if and only if one of the associated group operators is abelian.

(2.6) Example: Affine Space Let V be a vector space. Define

$$\lfloor,,\rfloor \Big| V \times V \times V \ni [x,y,z] \hookrightarrow x-y+z \in V\,, \tag{1}$$

which is a trinary libra operator on V . From a geometric point of view, $\lfloor x,y,z \rfloor$ is the unique element of V such that x , y , z and $\lfloor x,y,z \rfloor$ are adjacent vertices of a parallelogram.

(2.7) Definition Let L be a libra relative to a libra trinary operator $\lfloor,,\rfloor$. For $\{a,b\} \subset L$, we define the function

$$_a\pi_b \Big| L \ni x \hookrightarrow \lfloor a,x,b \rfloor \in L\,. \tag{1}$$

If $a = b$, or if L is abelian, then $_a\pi_b$ is an involution and called an **inner involution of the libra** L .

(2.8) Obverse Libras Let $\lfloor,,\rfloor$ be a libra trinary operator on a libra L. The **obverse of** $\lfloor,,\rfloor$ is the trinary operator $\lceil,,\rceil$ defined by

$$(\forall \{a,b,c\} \subset L) \quad \lceil a,b,c \rceil \equiv \lfloor c,b,a \rfloor . \tag{1}$$

In terms of the scales, a load θ is in equilibrium relative to obverse libra operator if, and only if,

$$\theta \circ \{[\heartsuit, \maltese],[\clubsuit, \bigstar]\} \tag{2}$$

is in equilibrium relative to the original libra operator. Thus for the obverse libra operator, the places on the scales are oriented in the opposite direction.

3. Libra Aggregates of Translates

(3.1) Balanced Subsets of a Libra A subset B of a libra L is said to be **balanced in** L if, whenever $\{x,y,z\}\subset B$, then $\lfloor x,y,z \rfloor$ is in B. Thus a balanced set B is a libra relative to the restriction of the trinary libra operator to $B\times B\times B$. Let B be a balanced subset of L. We introduce the notation

$$\lfloor r,s,B \rfloor \equiv \{\lfloor r,s,b \rfloor : b\in b\}, \quad \lfloor r,B,s \rfloor \equiv \{r,b,s : b\in B\} \quad \text{and} \quad \lfloor B,r,s \rfloor \equiv \{\lfloor b,r,s \rfloor : b\in b\}\,. \tag{1}$$

We call $\lfloor r,s,B \rfloor$ a **left translate of** B and $\lfloor B,r,s \rfloor$ a **right translate of** B. Sets of the form $\lfloor r,B,s \rfloor$ will be called, simply, **translates of** B.[8] The families of all left translates, right translates and translates of B, respectively, will be written

$$\boxed{B}\,, \quad \boxed{B} \quad \text{and} \quad \boxed{B}\,, \tag{2}$$

respectively. We adopt the notation

$$\boxed{B} \equiv \boxed{B} \cup \boxed{B}\,. \tag{3}$$

The elements of $\boxed{B}$ will be called **linear translations of** B and those of $\boxed{B} \triangle \boxed{B}$ **skew translations of** B.

(3.2) Theorem Let B be a balanced subset of a libra L. Then

$$(\forall\, b\in B) \quad B = \lfloor b,B,b \rfloor\,, \tag{1}$$

$$\boxed{B} \subset \boxed{B}\,, \tag{2}$$

and

$$\text{a translate of a translate of B is again a translate of B}\,. \tag{3}$$

Proof. That $\lfloor b,B,b \rfloor \subset B$ for $b\in B$ is trivial. Let $\{x,b\}\subset B$. Since B is balanced, we have $\lfloor b,B,b \rfloor \in B$. Thus

$$x \overset{\text{by (2.3.2)}}{=\!=} \lfloor b,b,x \rfloor \overset{\text{by (2.3.2)}}{=\!=} \lfloor \lfloor b,b,x \rfloor,b,b \rfloor \overset{\text{by (2.3.4)}}{=\!=} \lfloor b,\lfloor b,x,b \rfloor,b \rfloor \in \lfloor b,B,b \rfloor\,,$$

which implies that $B\subset \lfloor b,B,b \rfloor$. Hence (1) holds.

For $\{r,s\}\subset L$ and $b\in B$,

$$\lfloor r,s,B \rfloor \overset{\text{by (1)}}{=\!=} \lfloor r,s\lfloor b,B,b \rfloor \rfloor \overset{\text{by (2.3.3)}}{=\!=} \lfloor \lfloor r,s,b \rfloor,B,b \rfloor$$

and

$$\lfloor B,r,s \rfloor \overset{\text{by (1)}}{=\!=} \lfloor \lfloor b,B,b \rfloor,r,s \rfloor \overset{\text{by (2.3.3)}}{=\!=} \lfloor b,B,\lfloor b,r,s \rfloor \rfloor$$

which implies (2).

For $\{r,s,t,u\}\subset L$ we have, for any $b\in B$,

$$\lfloor r,\lfloor t,B,u \rfloor,s \rfloor \overset{\text{by (1)}}{=\!=} \lfloor r,\lfloor t,\lfloor b,B,b \rfloor,u \rfloor,s \rfloor \overset{\text{by (2.3.4)}}{=\!=} \lfloor r,\lfloor \lfloor t,b,B \rfloor,b,u \rfloor,s \rfloor \overset{\text{by (2.3.4)}}{=\!=}$$

$$\lfloor \lfloor r,u,b \rfloor,\lfloor t,b,B \rfloor,s \rfloor \overset{\text{by (2.3.4)}}{=\!=} \lfloor \lfloor r,u,b \rfloor,B,\lfloor b,t,s \rfloor \rfloor$$

which implies (3). Q.E.D.

---

8 We see in the following Theorem (2.2) that left and right translates are in fact translates in this sense.

(3.3) Definitions By a **homogeneous aggregate of translates**, or more simply, **aggregate**, we shall mean a family $\mathcal{T}$ of balanced sets, such that each element of $\mathcal{T}$ is a translate of each other element of $\mathcal{T}$, and such that each translate of a member of $\mathcal{T}$ is again a member of $\mathcal{T}$. It follows from Theorem (3.2) that the translates of any balanced set comprise an aggregate and that an aggregate is the family of translates of any one of its members:

$$(\forall\, \mathcal{T} \text{ an aggregate})(\forall\, \mathrm{T}\in\mathcal{T}) \quad \boxed{\mathrm{T}} = \mathcal{T}\,. \tag{1}$$

The family of all singletons is evidently a homogeneous aggregate of translates. We shall call it the **point aggregate of** L.

(3.4) Definition We shall say that a balanced set B is **normal** if each right translate of B is also a left translate of B.

(3.5) Lemma The following statements hold in any libra L:

$$(\forall\, \{a,b,c\}\subset L)(\exists!\, x\in L)\ a = \lfloor x,b,c\rfloor\,, \tag{1}$$

$$(\forall\, \{a,b,c\}\subset L)(\exists!\, x\in L)\ a = \lfloor b,x,c\rfloor\,, \tag{2}$$

$$(\forall\, \{a,b,c\}\subset L)(\exists!\, x\in L)\ a = \lfloor b,c,x\rfloor\,, \tag{3}$$

$$(\forall\, B\subset L \text{ balanced})(\forall\, \{b,y\}\subset L\colon\ b\in B) \quad \lfloor b,B,y\rfloor = \lfloor B,b,y\rfloor \tag{4}$$

and

$$(\forall\, B\subset L \text{ balanced})(\forall\, \{b,y\}\subset L\colon\ b\in B) \quad \lfloor y,b,B\rfloor = \lfloor y,B,b\rfloor\,. \tag{5}$$

Proof. $[\overset{(1)}{\Longrightarrow}]$ If x satisfies $a = \lfloor x,b,c\rfloor$, then

$$\lfloor a,c,b\rfloor = \lfloor \lfloor x,b,c\rfloor,c,b\rfloor \xlongequal{\text{by (2.3.3)}} \lfloor x,b,\lfloor c,c,b\rfloor\rfloor \xlongequal{\text{by (2.3.2)}} x$$

so there is only one element x of L for which $a = \lfloor x,b,c\rfloor$. That $x\equiv\lfloor a,c,b\rfloor$ satisfies (1) follows from direct computation.

$[\overset{(2)}{\Longrightarrow}]$ If x satisfies $a = \lfloor b,x,c\rfloor$, then

$$\lfloor c,a,b\rfloor = \lfloor c,\lfloor b,x,c\rfloor,b\rfloor \xlongequal{\text{by (2.3.4)}} \lfloor c,c,x,b,b\rfloor \xlongequal{\text{by (2.3.2)}} x$$

and so x is unique. That $x = \lfloor c,a,b\rfloor$ satisfies (2) follows from direct computation.

$[\overset{(3)}{\Longrightarrow}]$ Follows from an argument analogous to that used in showing (1).

$[\overset{(4)}{\Longrightarrow}]$ For $c\in B$

$$\lfloor b,c,y\rfloor \xlongequal{\text{by (2.3.2)}} \lfloor b,\lfloor b,b,c\rfloor,y\rfloor \xlongequal{\text{by (2.3.4)}} \lfloor\lfloor b,c,b\rfloor,b,y\rfloor \text{ which is in } \lfloor B,b,y\rfloor$$

and

$$\lfloor c,b,y\rfloor \xlongequal{\text{by (2.3.2)}} \lfloor\lfloor b,b,c\rfloor,b,y\rfloor \xlongequal{\text{by (2.3.4)}} \lfloor b,\lfloor b,c,b\rfloor,y\rfloor \text{ which is in } \lfloor b,B,y\rfloor$$

which implies (4).

$[\overset{(5)}{\Longrightarrow}]$ Follows by an argument analogous to that used in showing (4). Q.E.D.

(3.6) Theorem Let Be be a balanced subset of a libra L and let $\mathcal{T}$ be the smallest aggregate of which B is a member. Then the following statements are pairwise equivalent.

$$\text{B is normal,} \tag{1}$$

$$\text{each left translate of B is a right translate,} \tag{2}$$

$$\mathcal{T} \quad \text{is a partition of } L\,, \tag{3}$$

$$\boxed{\color{red}|\color{black}B} = \mathcal{T}\ , \tag{4}$$

$$\boxed{B\color{red}|} = \mathcal{T} \tag{5}$$

and

$$(\forall\, A \in \mathcal{T}) \quad A \text{ is normal.} \tag{6}$$

Proof. [(1)⟹(2)] Suppose that (1) holds, let $\{x,y\} \subset L$ and $b \in B$. From (1) follows that there exists $\{r,s\} \subset L$ such that

$$\lfloor\lfloor b,y,x\rfloor,b,B\rfloor = \lfloor B,r,s\rfloor\,. \tag{7}$$

We have

$$\begin{gathered}\lfloor b,\lfloor B,x,y\rfloor,b\rfloor \xlongequal{\text{by (2.3.4)}} \lfloor\lfloor b,y,x\rfloor,B,b\rfloor \xlongequal{\text{by (3.2.1)}} \lfloor\lfloor b,y,x\rfloor,\lfloor b,B,b\rfloor,b\rfloor \xlongequal{\text{by (2.3.4)}} \\ \lfloor\lfloor b,y,x\rfloor,b,\lfloor B,b,b\rfloor\rfloor \xlongequal{\text{by (2.3.2)}} \lfloor\lfloor b,y,x\rfloor,b,B\rfloor \xlongequal{\text{by (7)}} \lfloor B,r,s\rfloor\end{gathered} \tag{8}$$

and so

$$\begin{gathered}\lfloor B,x,y\rfloor \xlongequal{\text{by (2.3.2)}} \lfloor\lfloor b,b,\lfloor B,x,y\rfloor\rfloor,b,b\rfloor \xlongequal{\text{by (2.3.4)}} \lfloor b,\lfloor b,\lfloor B,x,y\rfloor,b\rfloor,b\rfloor \xlongequal{\text{by (8)}} \\ \lfloor b,\lfloor B,r,s\rfloor,b\rfloor \xlongequal{\text{by (2.3.4)}} \lfloor\lfloor b,s,r,\rfloor,B,b\rfloor \xlongequal{\text{by (3.2.1)}} \lfloor\lfloor b,s,r\rfloor,\lfloor b,B,b\rfloor,b\rfloor \xlongequal{\text{by (2.3.4)}} \\ \lfloor\lfloor b,s,r\rfloor,b,\lfloor b,b,B\rfloor\rfloor \xlongequal{\text{by (2.3.2)}} \lfloor\lfloor b,s,r\rfloor,b,B\rfloor\,.\end{gathered}$$

It follows that (2) holds.

[(2)⟹(3)] Suppose that (2) holds and assume that (3) does not. There would exist $\{A,C\} \subset \mathcal{T}$ such that $A \neq C$ and $A \cap C \neq \emptyset$. We could choose $\{u,v\} \subset L$ such that $B = \lfloor u,A,v\rfloor$ and let $D \equiv \lfloor u,C,v\rfloor$. Then $B \neq D$ and $B \cap C = \emptyset$. Choose b from $B \cap D$ and $d \in D$ such that $d \notin B$. Since $D = \lfloor d,b,B\rfloor$ is a left translate of B, (2) would imply that there exists $\{r,t\} \subset L$ such that $D = \lfloor B,r,t\rfloor$. Thus

$$d \in D = \lfloor B,r,t\rfloor \xlongequal{\text{by (2.3.2)}} \lfloor B,\lfloor r,b,b\rfloor,t\rfloor \xlongequal{\text{by (2.3.4)}} \lfloor B,b,\lfloor b,r,t\rfloor\rfloor \quad \text{and} \quad b \in D = \lfloor B,b,\lfloor r,r,t\rfloor\rfloor\,.$$

Thus we could choose $\{m,n\} \subset B$ such that $d = \lfloor m,b,\lfloor b,r,t\rfloor\rfloor$ and $b = \lfloor n,b,\lfloor b,r,t\rfloor\rfloor$. Then

$$\lfloor b,r,t\rfloor \xlongequal{\text{by (2.3.2)}} \lfloor\lfloor b,n,n\rfloor,\lfloor r,b,b\rfloor,t\rfloor \xlongequal{\text{by (2.3.4)}} \lfloor\lfloor b,n,n\rfloor,b\lfloor b,r,t\rfloor\rfloor \xlongequal{\text{by (2.3.3)}} \lfloor b,n,\lfloor n,b,\lfloor b,r,t\rfloor\rfloor\rfloor = \lfloor b,n,b\rfloor$$

which would imply that

$$d = \lfloor m,b,\lfloor b,r,t\rfloor\rfloor = \lfloor m,b,\lfloor b,n,b\rfloor\rfloor$$

whence would follow that d is in B. This would be absurd, so we have demonstrated that (3) holds.

[(3)⟹(4)] Suppose that (3) holds. Since $\boxed{\color{red}|\color{black}B}$ is a sub-family of $\mathcal{T}$ and $\boxed{\color{red}|\color{black}B}$ is also a partition, they must be identical partitions of L. Thus, (4) holds.

[(4)⟹(5)] Suppose that (4) holds. For $a \in \mathcal{T}$ and $b \in B$, it follows from (4) that there exists $\{x,y\} \subset L$ such that $\lfloor b,A,b\rfloor = \lfloor x,y,B\rfloor$. Thus

$$\begin{gathered}A \xlongequal{\text{by (2.3.2)}} \lfloor\lfloor b,b,A\rfloor,b,b\rfloor \xlongequal{\text{by (2.3.4)}} \lfloor b,\lfloor b,A,b\rfloor,b\rfloor = \lfloor b,\lfloor x,y,B\rfloor,b\rfloor \xlongequal{\text{by (2.3.4)}} \\ \lfloor\lfloor b,B,y\rfloor,x,b\rfloor \xlongequal{\text{by (3.5.4)}} \lfloor\lfloor B,b,y\rfloor,x,b\rfloor \xlongequal{\text{by (2.3.4)}} \lfloor B,b,\lfloor y,x,y\rfloor\rfloor \xlongequal{\text{by (2.3.2)}} \\ \lfloor\lfloor b,b,B\rfloor,b,\lfloor y,x,b\rfloor\rfloor \xlongequal{\text{by (2.3.4)}} \lfloor b,\lfloor b,B,b\rfloor,\lfloor y,x,b\rfloor\rfloor \xlongequal{\text{by (3.2.1) and by (3.5.4)}} \lfloor B,b,\lfloor y,x,b\rfloor\rfloor\,.\end{gathered}$$

which establishes (5).

[(5)⟹(6)] Suppose that (5) holds. For $A \in \mathcal{T}$ it follows from (5) that A is a right translate of B, whence follows that B is a right translate of A. Suppose that we had shown that each right translate of A is a left translate of B. Then A itself would be a left translate of B, and so B would be a left translate of B. Thus, to show that A is normal, it will suffice to show that, for $\{x,y\} \subset L$, $\lfloor A,x,y \rfloor$ must be a left translate of B. To this end we let b be any element of B and apply (5) to obtain $\{r,s\} \subset L$ such that $\lfloor \lfloor b,y,x \rfloor ,A,b \rfloor = \lfloor B,r,s \rfloor$. Then

$$\lfloor A,x,y \rfloor \overset{\text{by (2.3.2)}}{=\!=\!=} \lfloor \lfloor b,b,\lfloor A,x,y \rfloor \rfloor ,b,b \rfloor \overset{\text{by (2.3.4)}}{=\!=\!=} \lfloor b,\lfloor b,\lfloor A,x,y \rfloor ,b \rfloor ,b \rfloor \overset{\text{by (2.3.4)}}{=\!=\!=}$$

$$\lfloor b,\lfloor \lfloor b,y,x \rfloor ,A,b \rfloor ,b \rfloor = \lfloor b,\lfloor B,r,s \rfloor ,s \rfloor \overset{\text{by (2.3.4)}}{=\!=\!=} \lfloor \lfloor b,s,r \rfloor ,B,b \rfloor \overset{\text{by (3.5.4)}}{=\!=\!=} \lfloor \lfloor b,s,r \rfloor ,b,B \rfloor$$

which establishes (6).

That (6) implies (1) is trivial. Q.E.D.

(3.7) Normal Aggregates We shall say that a homogeneous aggregate of balanced sets is **normal** provided that all of its elements are normal balanced sets. From Theorem (3.6) follows that an aggregate is normal if, and only if, any one of its elements is normal.

Consequently, for a group G and a subgroup B, the aggregate of translates of B is normal if, and only if, B is a normal subgroup of G.

4. Function Libras and Representations

(4.1) Function Libras Just as permutation groups comprise the most important class of examples of abstract groups, function libras comprise the most important class of exambles of abstract libras. If X and Y are sets of equal cardinality, then the family $\mathrm{Y}^{\mathrm{X}}!$ of all bijections from X onto Y is a libra relative to the libra trinary operator

$$(\forall\,\{f,g,h\}\subset \mathrm{Y}^{\mathrm{X}}!)\quad [\![f,g,h]\!]\equiv f\circ g^{-1}\circ h\,. \tag{1}$$

A balanced subfamily of the libra $\mathrm{Y}^{\mathrm{X}}!$ is said to be a **function libra from** X **to** Y.

(4.2) Libra Isomorphisms A **libra homomorphism** $\phi|\mathrm{L}\hookrightarrow\mathrm{L}'$ from one libra L onto another libra L´ is a function such that

$$(\forall\,\{x,y,z\}\subset \mathrm{L})\quad \lfloor\phi(x),\phi(y),\phi(z)\rfloor'=\phi(\lfloor x,y,z\rfloor) \tag{1}$$

or, equivalently, if $\mathcal{E}$ denotes the equilibrium family of L, such that

$$(\forall\,\theta\in \mathrm{L}^{\mathrm{S}})\quad \theta\in\mathcal{E}\iff\phi\circ\theta\in\mathcal{E}'\,. \tag{2}$$

If a libra homomorphism is a bijection, we say that it is a **libra isomorphism**.

(4.3) Libra Multipliers By a **left multiplier of a libra** L we shall mean a function $\psi|\mathrm{L}\hookrightarrow\mathrm{L}$ such that

$$(\forall\,\{x,y,z\}\subset \mathrm{L})\quad \psi(\lfloor x,y,z\rfloor)=\lfloor\psi(x),y,z\rfloor\,. \tag{1}$$

If ψ is a left multiplier and $\{a,x\}\subset\mathrm{L}$, then

$$\psi(x)\;=\;\psi(\lfloor a,a,x\rfloor)=\lfloor\psi(a),a,x\rfloor\,. \tag{2}$$

It follows from (2.3.3) that

$$(\forall\,\{b,a\}\subset \mathrm{L})\quad \mathrm{L}\ni x\hookrightarrow\lfloor b,a,x\rfloor\in\mathrm{L}\text{ is a left multiplier} \tag{3}$$

and so it is a consequence of (2) that

$$\text{the left multipliers of L are just the functions of (3).} \tag{4}$$

A **right multiplier of a libra** L is a function $\omega|\mathrm{L}\hookrightarrow\mathrm{L}$ such that

$$(\forall\,\{x,y,z\}\subset \mathrm{L})\quad \psi(\lfloor x,y,z\rfloor)=\lfloor x,y,\psi(z)\rfloor\,. \tag{5}$$

As with left multipliers one can show that the family of right multipliers is just the family

$$\{\mathrm{L}\ni x\hookrightarrow\lfloor x,a,b\rfloor\in\mathrm{L}:\{a,b\}\subset\mathrm{L}\}\,. \tag{6}$$

(4.4) Libra Representations A libra homomorphism ρ from a libra L into a function libra (of bijections) from a set X onto a set Y will be called a **representation of** L. We adopt the notation

$$\lbrack\rho \equiv X\,, \quad \rho\rbrack \equiv Y \tag{1}$$

and

$$(\forall\, x \in \underline{\lfloor\rho\rfloor}) \quad \rho_x \equiv \rho(x)\,. \tag{2}$$

An injective representation is said to be **faithful**. If

$$(\forall\, [x,y] \in \lbrack\rho \times \rho\rbrack)(\exists\, a \in L) \quad \rho_a(x) = y\,, \tag{3}$$

we say that ρ is **homogeneous**.

For $[x,y] \in \lbrack\rho \times \rho\rbrack$, we shall frequently use the notation

$$[x \overset{\rho}{=} y] \equiv \{a \in \underline{\lfloor\rho\rfloor} : \rho_a(x) = y\}\,. \tag{4}$$

and

$$\mathcal{T}_\rho \equiv \{[x \overset{\rho}{=} y] : [x,y] \in \lbrack\rho \times \rho\rbrack\}\,. \tag{5}$$

Let $\mathcal{R}$ denote the family of right multipliers of a libra L. For $x \in L$, let

$$\rho_x \big| \mathcal{R} \ni \phi \hookrightarrow \phi(x) \in L\,. \tag{6}$$

For $\{t,y\} \subset L$ and $\omega \in \mathcal{R}$ such that $\omega(y) = t$, we have

$$(\forall\, s \in L) \quad \omega(s) = \omega(\lfloor s,y,y \rfloor) = \lfloor s,y,\omega(y) \rfloor = \lfloor s,y,t \rfloor$$

whence follows that

$$(\forall\, s \in L) \quad ({\rho_y}^{-1}(t))(s) = \lfloor s,y,t \rfloor\,. \tag{7}$$

From (7) follows that, for $\{x,y,z\} \subset L$ and $\phi \in \mathcal{R}$

$$[\![\rho_x,\rho_y,\rho_z]\!](\phi) = \rho_x \circ {\rho_y}^{-1}(\rho_z(\phi)) = ({\rho_y}^{-1}(\rho_z(\phi)))(x) = \lfloor x,y,\rho_z(\phi) \rfloor = \lfloor x,y,\phi(z) \rfloor = \phi(\lfloor x,y,z \rfloor) = \rho_{\lfloor x,y,z \rfloor}(\phi)\,.$$

This means that

$$\rho \text{ is a faithful representation of } L \text{ on the pair of sets } \mathcal{R} \text{ and } L\,. \tag{8}$$

This result (8) is an avatar for libras of Cayley's Theorem stating that each group is isomorphic to a group of permutations of itself.

(4.5) Theorem Let ρ be a homogeneous representation of a libra L. Then $\mathcal{T}_\rho$ is an aggregate of balanced sets.

Furthermore, the following statements are equivalent:

$$\mathcal{T}_\rho \text{ is normal}\,, \tag{1}$$

and

$$(\forall\, \{a,b\} \subset L) \quad \rho_a = \rho_b \Longleftrightarrow (\exists\, x \in \lbrack\rho) \ \rho_a(x) = \rho_b(x)\,. \tag{2}$$

Proof. Let T be an element of $\mathcal{T}_\rho$. Then there exists $[x,y] \in \lbrack\rho \times \rho\rbrack$ for which $T = [x \overset{\rho}{=} y]$. For $\{a,b\} \subset L$ and $u \in \lbrack\rho$ such that $\rho_b(u) = y$

$$(\forall\, c\in\mathcal{T}_\rho)\quad \rho_{\lfloor a,c,b\rfloor}(u)=\rho_a\circ\rho_c{}^{-1}(\rho_b(u))=\rho_a(x)\quad\Longrightarrow\quad \lfloor a,T,b\rfloor=[u\overset{\rho}{=}\rho_a(x)]\,. \tag{3}$$

Let S be another element of $\mathcal{T}_\rho$ and let $[w,z]\in\boxed{\rho}\times\boxed{\rho}$ such that $S=[w\overset{\rho}{=}z]$. Let $d\in L$ be such that $\rho_d(w)=y$ and choose $e\in L$ such that $\rho_e(x)=z$. Replacing T in (3) with S, a with d, and b with e, we obtain

$$\lfloor d,S,e\rfloor\;=\;[x\overset{\rho}{=}y]\;=\;T\,.$$

It follows that $\mathcal{T}_\rho$ is an aggregate.

[(1)⟹(2)] Suppose that (1) holds. For $\{a,b\}\subset L$ and $\{t,x\}\subset\boxed{\rho}$ suppose that $\rho_a(x)=\rho_b(x)$. Since $\mathcal{T}_\rho$ is normal, there exists $\{r,s\}\subset L$ such that $\lfloor r,s,[t\overset{\rho}{=}\rho_a(t)]\rfloor=[x\overset{\rho}{=}\rho_a(x)]$. Since a is in $[x\overset{\rho}{=}\rho_a(x)]$, there exists $c\in[t\overset{\rho}{=}\rho_a(t)]$ such that $a=\lfloor r,s,c\rfloor$. Consequently

$$\rho_a(t)=\rho_{\lfloor r,s,c\rfloor}(t)=\rho_r\circ\rho_s{}^{-1}(\rho_c(t))=\rho_r\circ\rho_s{}^{-1}(\rho_a(t))\,. \tag{4}$$

Since b is in $[t\overset{\rho}{=}\rho_a(t)]$, there exists $d\in[t\overset{\rho}{=}\rho_a(t)]$ such that $b=\lfloor r,s,d\rfloor$. Thus

$$\rho_b(t)=\rho_{\lfloor r,s,d\rfloor}(t)=\rho_r\circ\rho_s{}^{-1}(\rho_d(t))=\rho_r\circ\rho_s{}^{-1}(\rho_a(t))\overset{\text{by (4)}}{=\!=\!=}\rho_a(t)\,.$$

This establishes (2).

[(2)⟹(1)] Let B be an element of $\mathcal{T}_\rho$. There exists $[x,y]\in\boxed{\rho}\times\boxed{\rho}$ such that $B=[x\overset{\rho}{=}y]$. Let R be any right translate of B. There exists $\{r,q\}\subset L$ such that $R=\lfloor B,r,q\rfloor$. Since all the elements of B agree at x, it follows from (2) that they agree on $\rho_r\circ\rho_q{}^{-1}(x)$ as well — let $u\in\boxed{\rho}$ be this common value and let $v\equiv\rho_r{}^{-1}(u)$. Since ρ is a homogeneous representation, there exists $s\in L$ such that $\rho_s(v)=y$. For all $a\in B$

$$\rho_{\lfloor\lfloor s,r,a\rfloor,r,q\rfloor}(x)=\rho_s\circ\rho_r{}^{-1}\circ\rho_a\circ\rho_r{}^{-1}\circ\rho_q(x)=\rho_s\circ\rho_r{}^{-1}(u)=y\,.$$

It follows that B is normal. Q.E.D.

(4.6) Definition We shall say that a homogeneous representation is **normal** if either of the conditions (1) or (2) of Theorem (4.5) hold

(4.7) Theorem Let ρ be a normal homogeneous representation of a libra. Let $\{x,m\}\subset\boxed{\rho}$, $\{r,s,t\}\subset\boxed{\rho}$, $\{a,b,c,u,v,w\}\subset L$ and suppose

$$r=\rho_a(x)=\rho_u(m)\,,\quad s=\rho_b(x)=\rho_v(m)\quad\text{and}\quad t=\rho_c(x)=\rho_w(m)\,. \tag{1}$$

Then

$$\rho_{\lfloor a,b,c\rfloor}(x)=\rho_{\lfloor u,v,w\rfloor}(m)\,. \tag{2}$$

Proof. Let $\{d,e\}\subset L$ be such that $\rho_e(x)=\rho_d(m)$. Then

$$\rho_{\lfloor a,e,d\rfloor}(m)=r=\rho_u(m)\,,\quad \rho_{\lfloor b,e,d\rfloor}(m)=s=\rho_v(m)\quad\text{and}\quad \rho_{\lfloor c,e,d\rfloor}(m)=t=\rho_w(m)\,.$$

Since ρ is normal, it follows from (4.5.2) that $\rho_{\lfloor a,e,d\rfloor}=\rho_u$, $\rho_{\lfloor b,e,d\rfloor}=\rho_v$ and $\rho_{\lfloor c,e,d\rfloor}=\rho_w$. Thus

$$\rho_{\lfloor u,v,w\rfloor}(m)=\rho_{\lfloor\lfloor a,e,d\rfloor,\lfloor b,c,d\rfloor,\lfloor c,e,d\rfloor\rfloor}(m)=\rho_{\lfloor a,e,d,d,e,b,c,e,d\rfloor}(m)=\rho_{\lfloor a,b,c,e,d\rfloor}(m)=\rho_{\lfloor a,b,c\rfloor}(x)$$

which establishes (2). Q.E.D.

(4.8) Definition Let ρ be a normal homogeneous representation of a libra L. for $\{r,s,t\}\subset\boxed{\rho}$ we define

$$\lfloor r,s,t \rfloor_\rho \equiv \rho_{\lfloor a,b,c \rfloor}(x) \quad (\forall\, x \in \boxed{\rho} \text{ and } \{a,b,c\} \subset L\colon [r,s,t] = [\rho_a(x), \rho_b(x), \rho_c(x)]) . \tag{1}$$

In view of Theorem (4.7), $\lfloor ,, \rfloor_\rho$ is a well-defined libra operation on $\boxed{\rho}$. We shall call it the **repesentation libra operator induced on** $\boxed{\rho}$ by ρ.

(4.9) Theorem Let ρ be a faithful normal homogeneous representation of L. Then, for each $[x,y] \in \boxed{\rho} \times \boxed{\rho}$, the set $[x \overset{\rho}{=} y]$ is a singleton. In particular

$$(\forall\, x \in \boxed{\rho}) \quad L \ni a \hookrightarrow \rho_a(x) \in \boxed{\rho} \text{ is a bijection.} \tag{1}$$

Proof. Let $[x,y] \in \boxed{\rho}$. Assume that there were distinct elements a and b of $[x \overset{\rho}{=} y]$. Since ρ is faithful, there would exist $w \in \boxed{\rho}$ such that $\rho_a(w) \neq \rho_b(w)$. Since a would be in $[x \overset{\rho}{=} y] \cap [w \overset{\rho}{=} \rho_a(w)]$, it would follow from (3.6.3) that $[x \overset{\rho}{=} y] = [w \overset{\rho}{=} \rho_a(w)]$. Hence b would be in $[w \overset{\rho}{=} \rho_a(w)]$, which would be absurd. Q.E.D.

(4.10) Example: Line and Circle Let L be a line in the real projective plane $\mathbb{P}$, and let C be a circle in $\mathbb{P}$. Each point p of $L \triangle C$ induces a permutation p_C of C as in Figure (1). This family of permutations is a libra, and so induces a libra operator on L. The function p_C is actually a faithful normal homogeneous representation. That condition (4.5.2) holds is illustrated in the following figure:

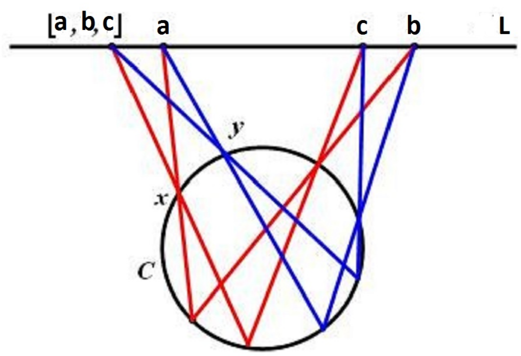


Fig. 7: Libra on the Line Induced by a Circle

The representation libra operator induced on C by the representation p_C is illustrated by the figure below:

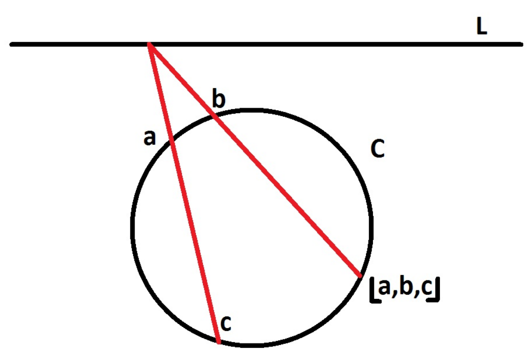


Fig. 8: Libra on the Circle Induced by a Line

(4.11) Definitions, Notation and Discussion In the sequel we shall often be concerned with representations which are not normal, and we shall treat these specifically in the next section. For this we shall need some definitions.

We recall that the obverse of a libra trinary operator $\lfloor,,\rfloor$ on a libra L is the trinary operator defined by

$$(\forall\,\{a,b,c\}\subset L)\quad \lceil a,b,c\rceil \equiv \lfloor c,b,a\rfloor\,. \tag{1}$$

If ρ is a representation of L, the **obverse $\tilde{\rho}$ of ρ** is the representation of L defined by

$$(\forall\,x\in L)\quad \tilde{\rho}_x \equiv {\rho_x}^{-1}\,. \tag{2}$$

The obverse representation is a representation of L *relative to the obverse operator* $\lceil,,\rceil$ *— not relative to the libra operator* $\lfloor,,\rfloor$.[9]

The **symmetrization of** L is the set L×L equipped with the **symmetrization operator** $\lfloor\!\lceil,,\rceil\!\rfloor$ defined by

$$(\forall\,\{[a,z],[b,y],[c,x]\}\subset L\times L)\quad \lfloor\!\lceil [a,z],[b,y],[c,x]\rceil\!\rfloor \equiv [\,\lfloor a,b,c\rfloor,\lceil z,y,x\rceil\,]\,. \tag{3}$$

The **symmetrization of the representation** ρ is the representation $\overset{\leftrightarrow}{\rho}$ defined by

$$(\forall\,[x,y]\in \boxed{\rho}\times\boxed{\rho})(\forall\,[a,b]\in L\times L)\quad \overset{\leftrightarrow}{\rho}_{[a,b]}([x,y]) \equiv [\,\tilde{\rho}_b(y),\rho_a(x)\,]\,. \tag{4}$$

Two representations ρ and θ will be said to be **equivalent** if there exist bijections $\mu\big|\boxed{\rho}\hookrightarrow\boxed{\theta}$ and $\nu\big|\boxed{\rho}\hookrightarrow\boxed{\theta}$ such that

$$(\forall\,a\in L)\quad \theta_a = \nu\circ\rho_a\circ\mu^{-1}\ :\ \textit{vid.}\quad \begin{array}{ccc} \boxed{\rho} & \xrightarrow{\ \rho_a\ } & \boxed{\rho} \\ \mu\downarrow & a\in L & \downarrow\nu \\ \boxed{\theta} & \xrightarrow{\ \theta_a\ } & \boxed{\theta} \end{array}\,. \tag{5}$$

[9] They are the same of course when L is abelian.

5. Cartesian Aggregates of Balanced Subsets of a Libra

(5.1) Columns and Rows of an Aggregate Let $\mathcal{T}$ be an aggregate of balanced subsets of a libra L. By a **row of** $\mathcal{T}$ we shall mean a sub-family $\mathcal{R}$ of $\mathcal{T}$ such that each element of $\mathcal{R}$ is a right translate of each other element of $\mathcal{R}$ and such that each right translate of an element of $\mathcal{R}$ is again an element of $\mathcal{R}$. By a **column of** $\mathcal{T}$ we shall mean a sub-family $\mathcal{C}$ of $\mathcal{T}$ such that each element of $\mathcal{C}$ is a left translate of each other element of $\mathcal{C}$ and such that each left translate of an element of $\mathcal{C}$ is again an element of $\mathcal{C}$. We shall write $\equiv(\mathcal{T})$ for the family of all rows of $\mathcal{T}$ and $||||(\mathcal{T})$ for the family of all columns of $\mathcal{T}$.

(5.2) Theorem For $\{a,b\}\subset L$, $[\mathcal{X},\mathcal{Y}]\in ||||(\mathcal{T})\times\equiv(\mathcal{T})$ and $[X,Y]\in\mathcal{X}\times\mathcal{Y}$

$$\{\lfloor a,W,b\rfloor : W\in\mathcal{Y}\} = \boxed{\lfloor a,Y,b\rfloor}\ , \tag{1}$$

and

$$\{\lfloor a,W,b\rfloor : W\in\mathcal{X}\} = \boxed{\lfloor a,X,b\rfloor}\ . \tag{2}$$

Proof. For $\{r,s\}\subset L$

$$\lfloor a,\lfloor Y,r,s\rfloor,b\rfloor = \lfloor\lfloor a,s,r\rfloor,Y,b\rfloor = \lfloor a,s,r,a,a,Y,b\rfloor = \lfloor\lfloor a,s,r\rfloor,a,\lfloor a,Y,b\rfloor\rfloor$$

whence follows that $\{\lfloor a,W,b\rfloor : W\in\mathcal{Y}\}\subset\boxed{\lfloor a,Y,b\rfloor}$. For $\{t,u\}\subset L$

$$\lfloor t,u,\lfloor a,Y,b\rfloor\rfloor = \lfloor a,a,t,u,a,Y,b\rfloor = \lfloor a,\lfloor u,t,a\rfloor,a,Y,b\rfloor = \lfloor a,\lfloor Y,a,\lfloor u,t,a\rfloor\rfloor,b\rfloor$$

which implies that $\boxed{\lfloor a,Y,b\rfloor}\subset\{\lfloor a,W,b\rfloor : W\in\mathcal{Y}\}$. It follows that (1) holds.

The proof of (2) is analogous to that of (1). Q.E.D.

(5.3) Notation Elements a and b of L produce bijections as follows

$$a\odot b\,\big|\,\mathcal{T}\ni B\hookrightarrow\lfloor a,B,b\rfloor\in\mathcal{T} \tag{1}$$

and

$$a\boxdot b\,\big|\,||||(\mathcal{T})\ni\mathcal{X}\hookrightarrow\{\lfloor a,X,b\rfloor : X\in\mathcal{X}\}\in\equiv(\mathcal{T})\ . \tag{2}$$

(5.4) Theorem For $b\in L$ and $\{r,s,t\}\subset L$

$$(s\odot b)^{-1} = b\odot s \tag{1}$$

and

$$\lfloor r,s,t\rfloor\odot b = \|r\odot b,s\odot b,t\odot b\|\ . \tag{2}$$

Proof. For $X\in\mathcal{T}$

$$b\odot s(s\odot b(X)) = \lfloor c,\lfloor s,X,c\rfloor,s\rfloor = \lfloor c,c,X,s,s\rfloor = X$$

which proves (1) and

$$(r\odot b)\circ(s\odot b)^{-1}\circ(t\odot b)(X) = (r\odot b)\circ(b\odot s)(\lfloor t,X,b\rfloor) = r\odot b(\lfloor b,\lfloor t,X,b\rfloor,s\rfloor) =$$

$$\lfloor r,\lfloor b,\lfloor t,X,b\rfloor,s\rfloor,b\rfloor = \lfloor r,\lfloor b,b,X,t,s\rfloor,b\rfloor = \lfloor r,s,t,X,b,b,b\rfloor = \lfloor\lfloor r,s,t\rfloor,X,b\rfloor = (\lfloor r,s,t\rfloor\odot b)(X)$$

which proves (2). Q.E.D.

(5.5) Notation For $a \in L$ we define

$$a^{\textcircled{\scriptsize T}} \equiv a \textcircled{\scriptsize T} a \quad \text{and} \quad a^{\boxed{\scriptsize T}} \equiv a \boxed{\scriptsize T} a\,. \tag{1}$$

(5.6) Theorem For $\{a,b,c\} \subset L$

$$\lfloor a,b,c \rfloor^{\boxed{\scriptsize T}} = \lfloor\!\lfloor a^{\boxed{\scriptsize T}}, b^{\boxed{\scriptsize T}}, c^{\boxed{\scriptsize T}} \rfloor\!\rfloor\,. \tag{1}$$

Proof. For $X \in \mathcal{T}$

$$a^{\boxed{\scriptsize T}} \circ b^{\boxed{\scriptsize T}-1} \circ c^{\boxed{\scriptsize T}} (X) = a^{\boxed{\scriptsize T}} \circ b^{\boxed{\scriptsize T}-1} \circ c^{\boxed{\scriptsize T}} (\lfloor \lfloor c,b,a \rfloor, \lfloor a,b,c \rfloor, X \rfloor) = a^{\boxed{\scriptsize T}} \circ b^{\boxed{\scriptsize T}-1} \circ c^{\boxed{\scriptsize T}} (\lfloor c,b,a,c,b,a,X \rfloor) =$$

$$a^{\boxed{\scriptsize T}} \circ b^{\boxed{\scriptsize T}-1} (\lfloor c,X,a,b,c,a,b,c,c \rfloor) = a^{\boxed{\scriptsize T}} \circ b^{\boxed{\scriptsize T}-1} (\lfloor c,X,a,b,c,a,b \rfloor) = a^{\boxed{\scriptsize T}} (\lfloor b,b,a,c,b,a,X,c,b \rfloor) =$$

$$a^{\boxed{\scriptsize T}} (\lfloor a,c,b,a,X,c,b \rfloor) = \lfloor a,b,c,X,a,b,c,a,a \rfloor = \lfloor \lfloor a,b,c \rfloor, X, \lfloor a,b,c \rfloor \rfloor = \lfloor a,b,c \rfloor^{\boxed{\scriptsize T}} (X)\,.$$

Q.E.D.

(5.7) Discussion and Notation The analogue of Theorem (5.6) for $\textcircled{\scriptsize T}$ does not hold and, in fact, a necessary and sufficient condition for $\lfloor a,b,c \rfloor^{\textcircled{\scriptsize T}}$ to equal $a^{\textcircled{\scriptsize T}} \circ b^{\textcircled{\scriptsize T}-1} \circ c^{\textcircled{\scriptsize T}}$ is for $\lfloor a,b,c \rfloor$ to equal $\lfloor c,b,a \rfloor$. It follows that in general, $\{a^{\textcircled{\scriptsize T}} : a \in L\}$ may not be a balanced set of the libra of permutations of $\mathcal{T}$. It is a simple computation to show that

$$(\forall \{a,b,m,n,r,s\} \subset L) \quad (a \textcircled{\scriptsize T} b) \circ (m \textcircled{\scriptsize T} n)^{-1} \circ (r \textcircled{\scriptsize T} s) = \lfloor a,m,r \rfloor \textcircled{\scriptsize T} \lfloor b,n,s \rfloor\,. \tag{1}$$

It follows that $\{x \textcircled{\scriptsize T} y : \{x,y\} \subset L\}$ is balanced. Since it contains $\{a^{\textcircled{\scriptsize T}} : a \in L\}$ as a subset, one may ask if it is the smallest balanced set containing this subset. We shall return to this question in Section (9) *infra.*

For $\{a,b,m,n\} \subset L$ we shall adopt the notation $\lfloor a,m \textcircled{\scriptsize T} n,b \rfloor$ for $(a \textcircled{\scriptsize T} b) \circ (n \textcircled{\scriptsize T} m)$:

$$\lfloor a,m \textcircled{\scriptsize T} n,b \rfloor \,\big|\, \mathcal{T} \ni A \hookrightarrow \{\lfloor a,m,x,n,b \rfloor : x \in A\} \in \mathcal{T}. \tag{2}$$

For $\{a,b,c,d,r,s,t,y\} \subset L$ and $A \in \mathcal{T}$, the computation

$$\lfloor a,b \textcircled{\scriptsize T} c,d \rfloor \circ \lfloor r,s \textcircled{\scriptsize T} t,u \rfloor (A) = \lfloor a,b \textcircled{\scriptsize T} c,d \rfloor (\lfloor r,s,A,t,u \rfloor) = \lfloor a,b,r,s,A,t,u,c,d \rfloor$$

shows that

$$\lfloor a,b \textcircled{\scriptsize T} c,d \rfloor \circ \lfloor r,s \textcircled{\scriptsize T} t,u \rfloor = \lfloor \lfloor a,b,r \rfloor, t \textcircled{\scriptsize T} s, \lfloor u,c,d \rfloor \rfloor = \lfloor a, \lfloor s,r,b \rfloor \textcircled{\scriptsize T} \lfloor c,u,t \rfloor, d \rfloor\,. \tag{3}$$

We shall adopt the notation

$$\mathfrak{Libra}(\mathcal{T}) \equiv \{a \textcircled{\scriptsize T} b : \{a,b\} \subset L\} \quad \text{and} \quad \mathfrak{Group}(\mathcal{T}) \equiv \{\lfloor a,b \textcircled{\scriptsize T} c,d \rfloor : \{a,b,c,d\} \subset L\}\,. \tag{4}$$

Theorem (5.6) implies that $\mathfrak{Libra}(\mathcal{T})$ is a libra. Furthermore, $\mathfrak{Group}(\mathcal{T})$ is a group since equality (3) implies that $\lfloor a,a \textcircled{\scriptsize T} a,a \rfloor$ is an identity for each $a \in L$ and that, for all $\{a,b,c,d\} \subset L$,

$$\lfloor a,b \textcircled{\scriptsize T} c,d \rfloor^{-1} = \lfloor b,a \textcircled{\scriptsize T} d,c \rfloor\,. \tag{5}$$

(5.8) Definitions Let ρ be a faithful representation of a libra L. We shall say that this representation is **cartesian** if, for any pair of doubletons $\{x,r\}\subset\boxed{\rho}$ and $\{y,s\}\subset\boxed{\rho}$,

$$(\exists\, a\in L)\quad \rho_a(x)=y \quad\text{and}\quad \rho_a(r)\neq s\,. \tag{1}$$

In particular, a cartesian representation is homogeneous.

Theorem (5.4) says that, for each $b\in L$, the function $L\ni a\hookrightarrow a\,\textcircled{\scriptsize $\mathcal{T}$}\,b\in\mathcal{T}^{\mathcal{T}}!$ is a representation of L. Theorem (5.6) says that the function sending each $a\in L$ to the restriction of $a^{\boxed{\mathcal{T}}}$ to $||||(\mathcal{T})$, is a representation of L. We shall call this latter the **left $\mathcal{T}$-inner representation of** L and denote it by

$$\lambda^{\mathcal{T}}. \tag{2}$$

(5.9) Theorem Let ρ be a cartesian representation of a libra L. Then ρ is equivalent to the left $\mathcal{T}_\rho$-inner representation. More precisely, for

$$\mu\Big|\boxed{\rho}\ni x\hookrightarrow\{[x\overset{\rho}{=}y]:y\in\boxed{\rho}\}\quad\text{and}\quad \nu\Big|\boxed{\rho}\ni y\hookrightarrow\{[x\overset{\rho}{=}y]:x\in\boxed{\rho}\} \tag{1}$$

then

$$(\forall\, a\in L)\quad a^{\boxed{\mathcal{T}}}=\nu\circ\rho_a\circ\mu^{-1}\quad : \textit{vid.}\quad \begin{array}{ccc} \boxed{\rho} & \xrightarrow{\;\rho_a\;} & \boxed{\rho} \\ \Big\downarrow\mu & a\in L & \Big\downarrow\nu \\ ||||(\mathcal{T}_\rho) & \xrightarrow{\;a^{\boxed{\mathcal{T}}}\;} & \equiv(\mathcal{T}_\rho) \end{array}\;. \tag{2}$$

Proof. We first show that, for each $x\in\boxed{\rho}$, $\mu(x)$ is in $||||(\mathcal{T}_\rho)$. Let y be in $\boxed{\rho}$ and $\{a,b\}\subset L$. For $k\equiv\rho_a\circ{\rho_b}^{-1}(y)$,

$$\lfloor a,b,[x\overset{\rho}{=}y]\rfloor=\{\lfloor a,b,t\rfloor:\rho_t(x)=y\}=\{w\in L:\rho_b\circ{\rho_a}^{-1}\circ\rho_w(x)=y\}=[x\overset{\rho}{=}k]\,.$$

Let s be another element of $\boxed{\rho}$. Let u be in $[x\overset{\rho}{=}y]$ and, exploiting the fact that ρ is cartesian, find $v\in L$ such that $\rho_v(x)=s$. That $\rho_u(x)=y$ implies that, for all $t\in L$ such that $\rho_t(x)=y$,

$$\rho_v\circ{\rho_u}^{-1}\circ\rho_t(x)=\rho_v(x)=s$$

whence follows that $\lfloor u,v,[x\overset{\rho}{=}y]\rfloor=[x\overset{\rho}{=}s]$. Consequently $\mu(x)$ is an element of $||||(\mathcal{T}_\rho)$.

That $\nu(y)$ is an element of $\equiv(\mathcal{T}_\rho)$ follows from an analogous argument. Q.E.D.

(5.10) Theorem Let ρ be as in Theorem (5.9). Then

$$(\forall\,\{a,b\}\subset L\text{ distinct})(\exists\, T\in\mathcal{T}_\rho)\quad a\in T\text{ and }b\notin T \tag{1}$$

and

$$(\forall\, T\in\mathcal{T}_\rho)(\forall\,\{a,b\}\subset L)\quad \lfloor a,T,b\rfloor=T\iff\{a,b\}\subset T\,. \tag{2}$$

Proof. $[\overset{(1)}{\Longrightarrow}]$ Since ρ is faithful, there exists $x\in\boxed{\rho}$ such that $\rho_a(x)\neq\rho_b(x)$. If $y\equiv\rho_a(x)$, then $a\in[x\overset{\rho}{=}y]$ but $b\notin[x\overset{\rho}{=}y]$.

$[\overset{(2)}{\Longrightarrow}]$ For $\{a,b\}\subset L$ it is trivial that $\lfloor a,T,b\rfloor=T$. We shall establish the reverse implication. Suppose that $\lfloor r,T,s\rfloor = T$ for $T \in \mathcal{T}_\rho$ and $\{r,s\} \subset L$. Then $\lfloor r,t,s\rfloor = u$ for $\{t,u\} \subset T$ and so $r = \lfloor u,s,t\rfloor$, whence follows that r must be in T if s is in T. Similarly, s must be in T if r is in T. Thus we may assume that neither r nor s is in T. Then we could choose $[x,y] \in \boxed{\rho} \times \boxed{\rho}$ such that $T = [x \overset{\rho}{=} y]$. Let $m \equiv \rho_r(x)$ and $y \equiv \rho_s{}^{-1}(y)$. Since ρ is cartesian, there would exist $t \in L$ such that $\rho_t(x) = y$ and $\rho_t(n) \neq m$. Then t would be in T and so in $\lfloor r,T,s\rfloor$ as well. Thus $t = \lfloor r,w,s\rfloor$ for $w \in T$, and so $w = \lfloor s,t,r\rfloor$. Consequently

$$y = \rho_w(x) = \rho_s \circ \rho_t{}^{-1} \circ \rho_r(x) = \rho_s \circ \rho_t{}^{-1}(m) \neq \rho_s(n) = y$$

which would be absurd. Q.E.D.

(5.11) Definition We shall say that an aggregate $\mathcal{T}$ is **cartesian** if both the conditions of Theorem (5.10) are satisfied:

$$(\forall \{a,b\} \subset L \text{ a doubleton})(\exists T \in \mathcal{T}) \quad a \in T \text{ and } b \notin T \tag{1}$$

and

$$(\forall T \in \mathcal{T})(\forall \{a,b\} \subset L) \quad T = \lfloor a,T,b\rfloor \iff \{a,b\} \subset T. \tag{2}$$

(5.12) Theorem Let $\mathcal{T}$ be a cartesian aggregate on a libra L. Then

$$(\forall \{a,b,c,d\} \subset L) \quad [a,b] = [c,d] \iff a \textcircled{$\scriptstyle\mathcal{T}$} b = c \textcircled{$\scriptstyle\mathcal{T}$} d. \tag{1}$$

Proof. There holds

$$a \textcircled{$\scriptstyle\mathcal{T}$} b = c \textcircled{$\scriptstyle\mathcal{T}$} d \iff (\forall X \in \mathcal{T}) \quad \lfloor a,X,b\rfloor = \lfloor c,X,d\rfloor \iff$$

$$(\forall X \in \mathcal{T}) \quad X = \lfloor b,b,X,a,a\rfloor = \lfloor b,\lfloor a,X,b\rfloor,a\rfloor = \lfloor b,\lfloor c,X,d\rfloor,a\rfloor = \lfloor b,d,X,c,a\rfloor \iff$$

$$(\forall X \in \mathcal{T},\ x \in S) \quad X = \lfloor b,d,x,X,x,c,a\rfloor \overset{\text{by } (5.10.2)}{\iff} (\forall X \in \mathcal{T},\ x \in X) \quad \{\lfloor b,d,x\rfloor,\lfloor x,c,a\rfloor\} \subset X. \tag{2}$$

If $a \neq c$, then (3.3.1) implies that there exists $C \in \mathcal{T}$ such that $c \in C$ and $a \notin C$. If $a \textcircled{$\scriptstyle\mathcal{T}$} b = c \textcircled{$\scriptstyle\mathcal{T}$} d$, then (2) would imply that were $y \in C$ such that $\lfloor c,c,a\rfloor = y$, which would in turn imply that $a = c$ which is in C. That would be absurd, whence follows that $a \textcircled{$\scriptstyle\mathcal{T}$} b \neq c \textcircled{$\scriptstyle\mathcal{T}$} d$. An analogous argument shows that if $b \neq c$, then $a \textcircled{$\scriptstyle\mathcal{T}$} b \neq c \textcircled{$\scriptstyle\mathcal{T}$} d$. Q.E.D.

(5.13) Theorem The following statements, regarding an aggregate of balanced subsets of a libra L, are pairwise equivalent:

$$(\forall B \in \mathcal{T})(\forall \{x,y\} \subset L) \quad \lfloor x,B,y\rfloor = B \iff \{x,y\} \subset B, \tag{1}$$

$$(\exists B \in \mathcal{T})(\forall \{x,y\} \subset L) \quad \lfloor x,B,y\rfloor = B \iff \{x,y\} \subset B, \tag{2}$$

$$(\exists B \in \mathcal{T}) \quad \boxed{B} \cap \boxed{B} = \{B\} \tag{3}$$

and

$$(\forall B \in \mathcal{T}) \quad \boxed{B} \cap \boxed{B} = \{B\}. \tag{4}$$

Proof [(1)⟹(2)] Trivial.

[(2)⟹(3)] Suppose that (2) holds for $B \in \mathcal{T}$. Suppose that $\lfloor r,s,B\rfloor = \lfloor B,t,u\rfloor$ for $\{r,s,t,u\} \in L$ and let b be an element of B. Letting $x \equiv \lfloor r,s,b\rfloor$ and $y = \lfloor b,t,u\rfloor$

$$\lfloor x,b,B\rfloor = \lfloor r,s,b,b,B\rfloor = \lfloor r,s,B\rfloor = \lfloor B,t,u\rfloor = \lfloor B,b,y\rfloor.$$

Thus

$$B = \lfloor b,x,x,b,B\rfloor = \lfloor b,x,\lfloor x,b,B\rfloor\rfloor = \lfloor b,x,\lfloor B,b,y\rfloor\rfloor = \lfloor b,x,\lfloor\lfloor b,B,b\rfloor,b,y\rfloor\rfloor =$$

$$\lfloor b,x,b,B,b,b,y \rfloor = \lfloor \lfloor b,x,b \rfloor ,B,y \rfloor .$$

From (2) follows that $\{y,\lfloor b,x,b \rfloor\} \subset B$.. Consequently $\lfloor r,s,B \rfloor = \lfloor x,b,B \rfloor = B$, and so $\boxed{B} \cup \boxed{B} = \{B\}$ .Hence (3) holds.

[(3)⟹(4)] Suppose that (3) holds for $B \in \mathcal{T}$ and that C is any other element of $\mathcal{T}$. There there exists $\{x,y\} \subset L$ such that $C = \lfloor x,B,y \rfloor$. Suppose that $\lfloor r,s,C \rfloor = \lfloor C,t,u \rfloor$ for $\{r,s,t,u\} \subset L$. Then

$$\lfloor B,\lfloor r,s,x \rfloor,x \rfloor = \lfloor y,y,B,x,s,r,x \rfloor = \lfloor y,\lfloor r,s,\lfloor x,B,y \rfloor \rfloor,x \rfloor = \lfloor y,\lfloor r,s,C \rfloor,x \rfloor =$$

$$\lfloor y,\lfloor C,u,t \rfloor,x \rfloor = \lfloor y,\lfloor \lfloor x,B,y \rfloor,u,t \rfloor,x \rfloor = \lfloor y,t,u,y,B,x,x \rfloor = \lfloor y,\lfloor y,u,t \rfloor,B \rfloor .$$

It follows from (3) that $\lfloor B,\lfloor r,s,x, \rfloor,x \rfloor = B$. Thus

$$C = \lfloor x,B,y \rfloor = \lfloor x,\lfloor B,\lfloor r,s,x \rfloor,x \rfloor,y \rfloor = \lfloor x,x,r,s,x,B,y \rfloor = \lfloor r,s,\lfloor x,B,y \rfloor \rfloor = \lfloor r,s,C \rfloor .$$

This implies (4).

[(4)⟹(1)] Suppose that (4) holds and let B be any element of $\mathcal{T}$. Suppose that $B = \lfloor x,B,y \rfloor$ for $\{x,y\} \subset L$. For $b \in B$

$$\lfloor b,x,B \rfloor = \lfloor b,x,\lfloor x,B,y \rfloor \rfloor = \lfloor \lfloor b,x,x \rfloor,B,y \rfloor = \lfloor \lfloor b,x,x \rfloor,\lfloor b,B,b \rfloor,y \rfloor = \lfloor b,x,x,b,B,b,y \rfloor = \lfloor B,b,y \rfloor .$$

From (4) follows that $\lfloor b,x,B \rfloor = B$. Hence $\lfloor b,x,b \rfloor = d$ for some $d \in B$, whence $x = \lfloor b,d,b \rfloor \in B$. It follows that $B = \lfloor b,x,B \rfloor = \lfloor B,b,y \rfloor$ which implies that $y = \lfloor b,b,y \rfloor \in \lfloor B,b,y \rfloor = B$. This means that (1) holds. QED

(5.14) Notation If two sets R and S have a singleton for their intersection, we shall denote the element of the singleton by $R \wedge S$:

$$\{R \wedge S\} \;=\; R \cap S . \tag{1}$$

(5.15) Theorem Let $\mathcal{T}$ be a cartesian aggregate on the libra L , let $\{\mathcal{A},\mathcal{B}\} \subset ||||(\mathcal{T})$ and let $\{\mathcal{C},\mathcal{D}\} \subset \equiv(\mathcal{T})$. Then

$$\mathcal{A} \wedge \mathcal{C} \text{ exists} \tag{1}$$

and

$$\mathcal{A} \wedge \mathcal{C} = \mathcal{B} \wedge \mathcal{D} \iff [\mathcal{A},\mathcal{C}] = [\mathcal{B},\mathcal{D}] . \tag{2}$$

Proof. [$\overset{(1)}{\Longrightarrow}$] Let A be in $\mathcal{A}$ and C be in $\mathcal{C}$. Since $\mathcal{T}$ is an aggregate of balanced sets, there exists $\{x,y\} \subset L$ such that $\lfloor x,A,y \rfloor = C$. For any $a \in A$

$$C = \lfloor x,A,y \rfloor = \lfloor x,\lfloor a,A,a \rfloor,y \rfloor = \lfloor \lfloor x,a,A \rfloor,a,y \rfloor$$

which implies

$$\lfloor C,y,a \rfloor = \lfloor x,a,A \rfloor$$

whence follows that $\mathcal{A} \cap \mathcal{C} \neq \emptyset$. That $\mathcal{A} \cap \mathcal{C}$ is a singleton follows from (5.13.4), which establishes (1).

[$\overset{(2)}{\Longrightarrow}$] Let $A \equiv \mathcal{A} \wedge \mathcal{C} = \mathcal{B} \wedge \mathcal{D}$. Then A is in $\mathcal{A}$ and $\mathcal{B}$ and so $\boxed{A} = \mathcal{A} = \mathcal{B}$. Similarly $\boxed{A} = \mathcal{C} = \mathcal{D}$. Thus $[\mathcal{A},\mathcal{C}] = [\mathcal{B},\mathcal{D}]$. The reverse implication of (2) is trivial. Q.E.D.

(5.16) Theorem Let $\mathcal{T}$ be a cartesian aggregate on the libra L . Then the left $\mathcal{T}$-inner representation is cartesian.

Proof. Let $\{x,y\}\subset L$ be a doubleton. Choose $B\in\mathcal{T}$ such that $x\in B$ and $y\notin B$. Then

$$x^{\boxed{\mathcal{T}}}(\boxed{B}) = \boxed{\lfloor x,B,x\rfloor} = \boxed{B}\,,$$

but for $b\in B$

$$y^{\boxed{\mathcal{T}}}(\boxed{B}) = u^{\boxed{\mathcal{T}}}(\boxed{\lfloor y,b,B\rfloor}) = \boxed{\lfloor y,\lfloor y,b,B\rfloor,y\rfloor} = \boxed{\lfloor\lfloor y,B,b\rfloor,y,y\rfloor} = \boxed{\lfloor y,B,b\rfloor} = \boxed{\lfloor y,b,B\rfloor}\,.$$

Since y is in $\lfloor y,b,B\rfloor$ but not in B, we know that $\lfloor y,b,B\rfloor\neq B$. It follows from Theorem (5.13.4) that $\boxed{B}\neq\boxed{\lfloor y,b,B\rfloor}$. Consequently $x^{\boxed{\mathcal{T}}}\neq y^{\boxed{\mathcal{T}}}$ and so the representation is faithful.

Let $\mathcal{A}$ and $\mathcal{B}$ be distinct elements of $||||(\mathcal{T})$ and $\mathcal{C}$ and $\mathcal{D}$ distinct elements of $\equiv(\mathcal{T})$. Choose a from $\mathcal{A}\wedge\mathcal{B}$ such that is not in $\mathcal{B}\wedge\mathcal{D}$. Then $a^{\boxed{\mathcal{T}}}(\mathcal{A})=\mathcal{C}$ but $a^{\boxed{\mathcal{T}}}(\mathcal{B})\neq\mathcal{D}$. Q.E.D.

(5.17) Discussion Theorems (5.9) and (5.16) imply that the cartesian aggregates of a libra L correspond exactly to the equivalence classes of cartesian representations of L. Along with the diagram of (5.9.1) we have its obverse:

$$(\forall\, a\in L)\quad \begin{array}{ccc} \boxed{\rho} & \xrightarrow{\ \rho_a\ } & \boxed{\rho} \\ \downarrow{\scriptstyle \mu} & a\in L & \downarrow{\scriptstyle \nu} \\ ||||(\mathcal{T}_\rho) & \xrightarrow{\ a^{\boxed{\mathcal{T}}}\ } & \equiv(\mathcal{T}_\rho) \end{array} \quad\text{and}\quad \begin{array}{ccc} \boxed{\rho} & \xleftarrow{\ \tilde{\rho}_a\ } & \boxed{\rho} \\ \downarrow{\scriptstyle \mu} & a\in L & \downarrow{\scriptstyle \nu} \\ ||||(\mathcal{T}_\rho) & \xleftarrow{\ a^{\boxed{\mathcal{T}}}\ } & \equiv(\mathcal{T}_\rho) \end{array}\,. \tag{1}$$

Each column $\mathcal{C}$ of $\mathcal{T}$ intersects each row $\mathcal{R}$ of $\mathcal{T}$ in exactly one element:

$$\{\mathcal{C}\wedge\mathcal{R}\}=\mathcal{C}\cap\mathcal{R}\,. \tag{2}$$

Thus $\wedge$ may be viewed as a bijection from $||||(\mathcal{T})\times\equiv(\mathcal{T})$ onto $\mathcal{T}$. The operator $\circledcirc_{\mathcal{T}}$ is actually a representation of the symmetrization libra $L\times L$ of L, and it is equivalent to the symmetrization $\overset{\leftrightarrow}{\rho}$ of the representation ρ:

$$(\forall\,[a,b]\in L\times L)\quad \begin{array}{ccc} \boxed{\rho}\times\boxed{\rho} & \xrightarrow{\ \overset{\leftrightarrow}{\rho}_{[a,b]}\ } & \boxed{\rho}\times\boxed{\rho} \\ \downarrow{\scriptstyle \mu\times\nu} & & \downarrow{\scriptstyle \mu\times\nu} \\ ||||(\mathcal{T}_\rho)\times\equiv(\mathcal{T}_\rho) & & ||||(\mathcal{T}_\rho)\times\equiv(\mathcal{T}_\rho) \\ \downarrow{\scriptstyle \wedge} & & \downarrow{\scriptstyle \wedge} \\ \mathcal{T} & \xrightarrow{\ a\circledcirc_{\mathcal{T}} b\ } & \mathcal{T} \end{array} \tag{3}$$

(where $\mu\times\nu([x,y])\equiv[\mu(x),\nu(y)]$). It is a corollary to Theorem (5.12) that

$$\text{the symmetrization representation } \overset{\leftrightarrow}{\rho} \text{ is faithful.} \tag{4}$$

The cardinality of $||||(\mathcal{T})$ is the same as the cardinality of $\equiv(\mathcal{T})$: we define the **dimension of** $\mathcal{T}$ to be that cardinal number. Thus the cardinality of $\mathcal{T}$ is the square of its dimension.

For $a \in L$ we define the **diagonal of $\mathcal{T}$ determined by** a to be

$$\diagdown\!a\!\diagdown \;\equiv\; \{A \in \mathcal{T} : a \in A\}\,. \tag{5}$$

The cardinality of such a diagonal is the same as the dimension of $\mathcal{T}$.

A diagonal $\diagdown\!a\!\diagdown$ can be used to give form to an aggregate in the sense that it associates to each column a row, and *vice versa*. A column $\mathcal{C}$ is a partition of L and so has exactly one element contained in $\diagdown\!a\!\diagdown$: this is the element $\mathcal{C} \wedge \diagdown\!a\!\diagdown$. We have the bijections

$$||||(\mathcal{T}) \ni \mathcal{C} \;\hookrightarrow\; \boxed{\mathcal{C} \wedge \diagdown\!a\!\diagdown} \in \equiv(\mathcal{T}) \quad \text{and} \quad \equiv(\mathcal{T}) \ni \mathcal{R} \;\hookrightarrow\; \boxed{\mathcal{R} \wedge \diagdown\!a\!\diagdown} \in ||||(\mathcal{T})\,. \tag{6}$$

If $\{A_i : i \in N\}$ is a well ordering of $\diagdown\!a\!\diagdown$, then the aggregate $\mathcal{T}$ may be visualized as the elements of a matrix:

$$\begin{pmatrix} A_1 & \boxed{A_1} \wedge \boxed{A_2} & \cdots & \boxed{A_1} \wedge \boxed{A_n} & \cdots \\ \boxed{A_2} \wedge \boxed{A_1} & A_2 & \cdots & \boxed{A_2} \wedge \boxed{A_n} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ \boxed{A_n} \wedge \boxed{A_n} & \boxed{A_n} \wedge \boxed{A_2} & \cdots & A_n & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \tag{7}$$

Once an aggregate is visualized as a matrix, one can depict the actions of the operators $x \circledcirc y$ and $x^{\circledcirc}$ for $\{x,y\} \subset L$. Let for instance B be an element of $\mathcal{T}$ and b an element of B. Then x is in $\lfloor x,b,B \rfloor$ and y in $\lfloor B,b,y \rfloor$ and so

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \lfloor B,b,y \rfloor & \cdots & B & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & x \circledcirc y\,(B) & \cdots & \lfloor x,b,B \rfloor & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \lfloor B,b,x \rfloor & \cdots & B & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & x^{\circledcirc}(B) & \cdots & \lfloor x,b,B \rfloor & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{8}$$

In the case $x = a$, there exists $\{i,j\} \subset N$ such that $\lfloor B,b,a \rfloor = A_i$ and $\lfloor a,b,B \rfloor = A_j$ and so the second matrix of (8) becomes

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & A_i & \cdots & B & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & a^{\circledcirc}(B) & \cdots & A_j & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{9}$$

We say then that B and $a^{\circledcirc}(B)$ are symmetric relative to $\diagdown\!a\!\diagdown$. More formally, two elements B and C of $\mathcal{T}$ are **symmetric relative to the diagonal of $\mathcal{T}$ determined by** a if $C = \lfloor a,B,a \rfloor$.

We shall say that B and C in $\mathcal{T}$ are **skew** provided that

$$\boxed{B} \neq \boxed{C} \quad \text{and} \quad \boxed{C} \neq \boxed{B}\,. \tag{10}$$

We shall say that B and C in $\mathcal{T}$ are a-**skew** provided that they are skew and that they are not symmetric relative to a.

(5.18) Theorem Let x be in a libra L and let $\mathcal{T}$ be a cartesian aggregate for L. Then

$$(\forall \{A,B\} \subset \mathcal{T}) \quad x^{\textcircled{\mathcal{T}}}(A) = B \iff x \in (\boxed{A} \wedge \boxed{B}) \cap (\boxed{B} \wedge \boxed{A}) \tag{1}$$

and

$$(\forall A \in \mathcal{T}) \quad x^{\textcircled{\mathcal{T}}}(A) = A \iff x \in A \iff x^{\boxed{\mathcal{T}}}(\boxed{A}) = \boxed{A}\,. \tag{2}$$

Proof. $[\overset{(1)}{\Longrightarrow}]$ Suppose that $x^{\textcircled{\mathcal{T}}}(A) = B$ for $\{A,B\} \subset \mathcal{T}$. Let a be an element of A. Then

$$B = x^{\textcircled{\mathcal{T}}}(A) = \lfloor x,A,x \rfloor = \lfloor x, \lfloor a,A,a \rfloor, x \rfloor = \lfloor \lfloor x,a,A \rfloor, a,x \rfloor \implies \lfloor x,a,A \rfloor = \lfloor B,x,a \rfloor\,. \tag{3}$$

Thus

$$x = \lfloor x,a,a \rfloor \in \lfloor x,a,A \rfloor \overset{\text{by (1)}}{=\!=\!=} \boxed{A} \wedge \boxed{B}\,. \tag{4}$$

Now let b be an element of B. Then

$$A = x^{\textcircled{\mathcal{T}}} \circ x^{\textcircled{\mathcal{T}}}(A) = x^{\textcircled{\mathcal{T}}}(B) = \lfloor x,B,x \rfloor = \lfloor x,b,B,b,x \rfloor = \lfloor \lfloor x,b,B \rfloor, b,x \rfloor \implies \lfloor A,x,b \rfloor = \lfloor x,b,B \rfloor\,.$$

This implies that $x = \lfloor x,b,b \rfloor \in \lfloor x,b,B \rfloor = \boxed{B} \wedge \boxed{A}$ which, with (4) yields $x \in (\boxed{A} \wedge \boxed{B}) \cap (\boxed{B} \wedge \boxed{A})$.

$[\overset{(1)}{\Longleftarrow}]$ Suppose that x is in $(\boxed{A} \wedge \boxed{B}) \cap (\boxed{B} \wedge \boxed{A})$. Let a be in A and b in B. Then x is in $\boxed{A} \wedge \boxed{B}$ which implies that $\lfloor x,a,A \rfloor = \lfloor B,b,x \rfloor$. That x is in $\boxed{B} \wedge \boxed{A}$ implies that $\lfloor A,a,x \rfloor = \lfloor x,b,B \rfloor$. Hence

$$\lfloor B,b,\lfloor x,a,x \rfloor \rfloor = \lfloor \lfloor B,b,x \rfloor, a,x \rfloor = \lfloor \lfloor x,a,A \rfloor, a,x \rfloor = \lfloor x,a,\lfloor A,a,x \rfloor \rfloor = \lfloor x,a,\lfloor x,b,B \rfloor \rfloor = \lfloor \lfloor x,a, \rfloor, b,B \rfloor\,.$$

The only right translate of B which is also a left translate of B is B itself. Thus

$$B = \lfloor B,b,\lfloor x,a,x \rfloor \rfloor \implies (\exists \{c,d\} \subset B) \quad c = \lfloor d,b,\lfloor x,a,a \rfloor \rfloor \implies \lfloor x,a,x \rfloor = \lfloor b,d,c \rfloor \in B\,.$$

Consequently

$$x^{\textcircled{\mathcal{T}}}(A) = \lfloor x,A,x \rfloor \subset B \implies \lfloor x,a,x \rfloor = \lfloor b,d,c \rfloor \in B$$

which establishes (1).

When $A = B$ we have $(\boxed{A} \wedge \boxed{B}) \cap (\boxed{B} \wedge \boxed{A}) = A \cap A = A$. Thus the first $\iff$ of (2) is a special case of (1). For x in X

$$x^{\boxed{\mathcal{T}}}(\boxed{A}) = \boxed{\lfloor x,A,x \rfloor} = \boxed{A}\,.$$

Suppose, on the other hand, that $x^{\boxed{\mathcal{T}}}(\boxed{A}) = A$, and let a be any element of A. Evidently $x \in \lfloor x,a,A \rfloor$ and so

$$x = \lfloor x,x,x \rfloor \in \lfloor x,\lfloor x,a,A \rfloor,x \rfloor = \lfloor x,A,a,x,x \rfloor = \lfloor x,a,a,A,a \rfloor = \lfloor x,a,A \rfloor\,.$$

Thus $\lfloor x,\lfloor x,a,A \rfloor,x \rfloor = x^{\textcircled{\mathcal{T}}}(\lfloor x,a,A \rfloor)$ is both a right translate and a left translate of A, whence follows it must be A. Hence $x \in \lfloor x,a,A \rfloor = A$. This finishes the proof of the second "$\iff$" of (2). Q.E.D.

(5.19) Notation Let $\mathcal{X}$ be in $⦀(\mathcal{T}) \cup \equiv(\mathcal{T})$ and let x be in L. Since $\mathcal{X}$ is a partition of L, we may define

$$x \wedge \mathcal{X} \equiv Y \quad \text{for } x \in Y \text{ and } Y \in \mathcal{X}. \tag{1}$$

(5.20) Theorem Let $\mathcal{T}$ be a cartesian aggregate for a libra L and let $\{a,b\} \subset L$. Then $a \boxed{\mathcal{T}} b$ agrees with $a^{\boxed{\mathcal{T}}}$ on $\|\|\|\|(\mathcal{T})$ and agrees with $b^{\boxed{\mathcal{T}}}$ on $\equiv(\mathcal{T})$.

Proof. For $T\in\mathcal{T}$ and $t\in T$

$$\{\lfloor a,\lfloor m,t,T\rfloor,a\rfloor : m\in L\} = \{\lfloor a,T,\lfloor t,m,a\rfloor\rfloor : m\in L\} = \{\lfloor a,T,n\rfloor : n\in L\} =$$

$$\{\lfloor a,T,\lfloor t,m,b\rfloor\rfloor : m\in L\} = \{\lfloor a,\lfloor m,t,T\rfloor,b\rfloor : m\in L\} = a\boxed{\mathcal{T}}b(\,[\boxed{T}]\,)$$

and

$$\{\lfloor b,\lfloor T,t,m\rfloor,b\rfloor : m\in L\} = \{\lfloor\lfloor b,m,t\rfloor,t,b\rfloor : m\in L\} = \{\lfloor n,T,b\rfloor : n\in L\} =$$

$$\{\lfloor\lfloor a,m,t\rfloor,T,b\rfloor : m\in L\} = \{\lfloor a,\lfloor T,t,m\rfloor,b\rfloor : m\in L\} = a\boxed{\mathcal{T}}b(\,[\![\boxed{T}]\!]\,) .$$

Q.E.D.

6. Libra Incidence

(6.1) Incidence Relations A function libra from one set X to another Y is sometimes subject to a curious involution when there exists what is called an "incidence relation between X and Y".

Let $\Bumpeq$ be any non-void graph.[10] We introduce the notation

$$(\forall\,[A,B]\in\lfloor\Bumpeq\rfloor\times\lceil\Bumpeq\rceil)\quad A^{o}\equiv\{y\in\lceil\Bumpeq\rceil:(\forall\,a\in A)\ \ [a,y]\subset\Bumpeq\}\quad\text{and}\quad B^{o}\equiv\{x\in\lfloor\Bumpeq\rfloor:(\forall\,b\in B)\ \ [x,b]\subset\Bumpeq\}, \tag{1}$$

$$A^{oo}\equiv(A^{o})^{o}\quad\text{and}\quad B^{oo}\equiv(B^{o})^{o}, \tag{2}$$

$$A^{ooo}\equiv(A^{oo})^{o}\quad\text{and}\quad B^{ooo}\equiv(B^{oo})^{o}. \tag{3}$$

It is evident that, if $A'\subset A\subset\lfloor\Bumpeq\rfloor$ and $B'\subset B\subset\lceil\Bumpeq\rceil$, then

$$A^{o}\subset A'^{o}\quad\text{and}\quad B^{o}\subset B'^{o} \tag{4}$$

and so

$$(\forall\,A\subset\lfloor\Bumpeq\rfloor\text{ and }B\subset\lceil\Bumpeq\rceil)\quad A^{ooo}=A^{o}\quad\text{and}\quad B^{ooo}=B^{o}. \tag{5}$$

For $A\subset\lfloor\Bumpeq\rfloor$ and $B\subset\lceil\Bumpeq\rceil$ we say that A^{oo} is the **span of** A and that B^{oo} is the **span of** B **relative to the graph** $\Bumpeq$. When dealing with singletons $\{s\}$ for $s\in\lfloor\Bumpeq\rfloor\cup\lceil\Bumpeq\rceil$ we will at times employ the notation

$$s^{\diamond}\equiv\{s\}^{o},\ s^{\diamond o}\equiv\{s\}^{oo}\quad\text{and}\quad s^{\diamond oo}\equiv\{s\}^{oo}. \tag{6}$$

The notation

$$x\Bumpeq y \tag{7}$$

will be taken to have the same meaning as $[x,y]\in\Bumpeq$.

The graph $\Bumpeq\subset\lfloor\Bumpeq\rfloor\times\lceil\Bumpeq\rceil$ will be said to be an **incidence relation from** $\lfloor\Bumpeq\rfloor$ **to** $\lceil\Bumpeq\rceil$ if

$$(\forall\,[x,y]\in\lfloor\Bumpeq\rfloor\times\lceil\Bumpeq\rceil)\quad\{x\}=x^{\diamond o}\quad\text{and}\quad\{y\}=y^{\diamond o}. \tag{8}$$

A necessary and sufficient condition for a graph $\Bumpeq$ to be an incidence relation is that both

$$(\forall\,\{x,y\}\subset\lfloor\Bumpeq\rfloor\text{ a doubleton})(\exists\,z\in\lceil\Bumpeq\rceil)\quad[x,z]\in\Bumpeq\text{ and }[y,z]\notin\Bumpeq \tag{9}$$

and

$$(\forall\,\{x,y\}\subset\lceil\Bumpeq\rceil\text{ a doubleton})(\exists\,z\in\lfloor\Bumpeq\rfloor)\quad[z,x]\in\Bumpeq\text{ and }[z,y]\notin\Bumpeq. \tag{10}$$

A direct consequence of (9) is that, for an incidence relation $\Bumpeq$,

$$\lfloor\Bumpeq\rfloor^{o}=\emptyset=\lceil\Bumpeq\rceil^{o}. \tag{11}$$

For a doubleton $\{a,b\}$ either in $\lfloor\Bumpeq\rfloor$ or in $\lceil\Bumpeq\rceil$, we adopt the notation

$$\overleftrightarrow{a,b}\equiv\{a,b\}^{oo}. \tag{12}$$

If

[10] By a graph we mean here a "relation" or a subset of a cartesian product of sets.

$$(\forall\,\{x,y\}\subset \sqcup_{\asymp})(\exists\, z\in \sqcap^{\asymp})\quad x\asymp z \text{ and } y\asymp z\,, \tag{13}$$

and

$$(\forall\,\{x,y\}\subset \sqcap^{\asymp})(\exists\, z\in \sqcup_{\asymp})\quad z\asymp x \text{ and } z\asymp y\,, \tag{14}$$

we shall say that $\asymp$ is a **properly linear incidence relation**. An incidence relation $\asymp$ is properly linear if, and only if

$$(\forall\,\{a,b\}\subset \sqcup_{\asymp} \text{ a doubleton})\quad \overleftrightarrow{a,b}\neq \sqcup_{\asymp} \tag{15}$$

and

$$(\forall\,\{a,b\}\subset \sqcap^{\asymp} \text{ a doubleton})\quad \overleftrightarrow{a,b}\neq \sqcap^{\asymp}\,. \tag{16}$$

We shall call the function $(2^{\sqcup_{\asymp}}\cup 2^{\sqcap^{\asymp}})\ni X\hookrightarrow X^{\circ}\in(2^{\sqcup_{\asymp}}\cup 2^{\sqcap^{\asymp}})$ the **exclusive set function associated with** $\asymp$.

(6.2) Libra Twining Relation Let $\lfloor,,\rfloor$ be a libra trinary operation on a libra L. The **libra twining relation** is the graph

$$\overset{L}{\asymp}\ \equiv\ \{[x,y]\in L\times L : \lfloor x,y,x\rfloor = y \quad\text{and}\quad x\neq y\}\,. \tag{1}$$

In this context we shall replace the associated exclusive set function symbols A° and $x^{\diamond}$, respectively, by

$$A^{\bullet} \text{ and } x^{\diamond}\,, \tag{2}$$

respectively.

If the libra twining relation is properly linear, we shall say that L is a **properly linear libra**.

(6.3) Theorem Let B be a balanced subset of a libra L and let A be a non-void subset of L. Then $B\cap A^{\bullet}$ is balanced if, and only if, it is abelian.

Proof. For $\{a,b,c\}\subset B\cap A^{\bullet}$ and $i\in A$

$$\lfloor i,\lfloor a,b,c\rfloor,i\rfloor = \lfloor i,c,b,a,i\rfloor = \lfloor i,c,i,i,b,i,i,a,i\rfloor = \lfloor\lfloor i,c,i\rfloor,\lfloor i,b,i\rfloor,\lfloor i,a,i\rfloor\rfloor = \lfloor c,b,a\rfloor$$

which implies that $\lfloor a,b,c\rfloor$ is in $i^{\diamond}$ precisely if $\lfloor a,b,c\rfloor = \lfloor c,b,a\rfloor$. Q.E.D.

(6.4) Theorem Let L be a libra and let A and B be balanced span subsets of L. Then

$$A = B^{\bullet} \iff B = A^{\bullet}\,. \tag{1}$$

Furthermore, if (1) holds, then

$$A\cup B \text{ is balanced.} \tag{2}$$

Proof. Suppose that $A=B^{\bullet}$. Then $A^{\bullet}=B^{\bullet\bullet}$. Since B is a span, it follows that $B=B^{\bullet\bullet}$. Hence $B=A^{\bullet}$. That $B=A^{\bullet}$ implies $A=B^{\bullet}$ follows from an analogous argument.

We now suppose that $A=B^{\bullet}$. Let $\{x,y,z\}\subset A\cup B$. If $\{x,y,z\}\subset A$ or $\{x,y,z\}\subset B$, then $\lfloor x,y,z\rfloor$ would be in $A\cup B$ since both A and B are balanced. Thus, without loss of generality, we can and shall suppose that $\{x,y\}\subset A$ and $z\in B$. We need to show that $\{\lfloor x,y,z\rfloor,\lfloor x,z,y\rfloor,\lfloor z,x,y\rfloor\}\subset A\cup B$ — but since z is in $A^{\bullet}$ we see that all three are the same. We see

$$\lfloor a,\lfloor x,y,z\rfloor,a\rfloor = \lfloor a,z,\lfloor y,x,z\rfloor\rfloor = \lfloor a,z,\lfloor a,x,y\rfloor\rfloor = \lfloor a,z,a,x,y\rfloor =$$
$$\lfloor a,a,z,x,y\rfloor = \lfloor a,a,x,z,y\rfloor = \lfloor a,a,x,y,z\rfloor = \lfloor x,y,z\rfloor .$$

It follows that $\lfloor x,y,z\rfloor$ is in $A^{\bullet} = B \subset (A\cup B)$. It follows that $A\cup B$ is balanced.Q.E.D.

(6.5) Theorem Let A and B be spans and such that $A = B^{\bullet}$. Then

$$(\forall\, [a,b] \in A\times B) \quad B = \lfloor a,b,A\rfloor . \tag{1}$$

Proof. From (6.4) follows that $A\cup B$ is abelian. Let $\{a',a''\}$ be in A. We have

$$\lfloor a'',\lfloor a,b,a'\rfloor,a''\rfloor = \lfloor a'',a',b,a,a''\rfloor = \lfloor a'',a',a,a,b,a,a''\rfloor = \lfloor a'',a',a,\lfloor a,b,a\rfloor,a''\rfloor = \lfloor a,a',a'',b,a''\rfloor =$$
$$\lfloor a'',a',a,a,b,a,a''\rfloor = \lfloor a'',a',a,\lfloor a,b,a\rfloor,a''\rfloor = \lfloor a,a',b\rfloor = \lfloor a,a',b,a',a'\rfloor = \lfloor a,\lfloor a',b,a'\rfloor,a'\rfloor = \lfloor a,b,a'\rfloor .$$

This shows that $\lfloor a,b,A\rfloor$ is a subset $A^{\bullet}$: thus

$$\lfloor a,b,A\rfloor \subset B . \tag{2}$$

On the other hand,

$$b = \lfloor a,b,a\rfloor \in \lfloor a,b,A\rfloor$$

which implies

$$B \subset \lfloor a,b,A\rfloor .$$

This, with (2), implies that $B = \lfloor a,b,A\rfloor$ .Q.E.D.

(6.6) Definition We shall say that a libra L is a **canopy** if there exists $a\in L$ such that

$$(\forall\, B\subset L \text{ balanced}:\ a^{\blacklozenge}\subset B) \quad B = L . \tag{1}$$

(6.7) Theorem If a libra L is a canopy, then

$$(\forall\, c\in L)(\forall\, B\subset L \text{ balanced}:\ c^{\blacklozenge}\subset B) \quad B = L . \tag{1}$$

Proof. Let $a\in L$ be such that (6.6.1) holds. Let c be in L and let B be any balanced set in L for which $c^{\blacklozenge}\subset B$. If x is an element of $a^{\blacklozenge}$, then

$$\lfloor\lfloor x,a,b\rfloor,b,\lfloor x,a,b\rfloor\rfloor = \lfloor\lfloor x,a,x\rfloor,a,b\rfloor = \lfloor a,a,b\rfloor = b \implies \lfloor x,a,b\rfloor \in b^{\blacklozenge} \implies \lfloor x,a,b\rfloor \in B \implies x\in \lfloor B,b,a\rfloor .$$

It follows that $a^{\blacklozenge}\subset \lfloor B,b,a\rfloor$, whence follows that $L = \lfloor B,b,a\rfloor$, which implies that $L = B$. Q.E.D.

(6.8) Theorem Let L be a canopy libra and let $\{u,v\}\subset L$. Then there exists $n\in\mathbb{N}$ odd and $\{x_1,\ldots,x_n\}\subset u^{\blacklozenge}$ such that

$$\lfloor x_1,\ldots,x_n\rfloor = v . \tag{1}$$

Proof. Let a be as in (6.6.1). Since no proper balanced subset of L contains $a^{\blacklozenge}$ as a subset, and since

$$\{\lfloor t_1,\ldots,t_n\rfloor : n\in\mathbb{N} \text{ is odd and } \{t_1,\ldots,t_n\}\subset a^{\blacklozenge}\}$$

is a balanced subset of L containing $a^{\blacklozenge}$, there exists $n\in\mathbb{N}$ odd and

$$\{t_1,\ldots,t_n\}\subset a^{\blacklozenge} \tag{2}$$

such that

$$\lfloor t_1,\ldots,t_n\rfloor = \lfloor a,u,v\rfloor\,. \tag{3}$$

For $i=1,\ldots,n$ and $x_i \equiv \lfloor u,a,t_i\rfloor$

$$\lfloor x_1,\ldots,x_n\rfloor = \lfloor \lfloor u,a,t_1\rfloor,\lfloor u,a,t_2\rfloor,\lfloor u,a,t_3\rfloor,\ldots,\lfloor u,a,t_n\rfloor\rfloor =$$

$$\lfloor u,a,t_1,t_2,a,u,u,a,t_3,\ldots,u,a,t_n\rfloor = \lfloor u,a,t_1,t_2t,\ldots t_n\rfloor = \lfloor u,a,\lfloor a,u,v\rfloor\rfloor = v\,.$$

Moreover, for each i

$$\lfloor u,x_i,u\rfloor = \lfloor u,\lfloor u,a,t_i\rfloor,u\rfloor = \lfloor u,t_i,a,u,u\rfloor = \lfloor u,t_i,a\rfloor \overset{\text{by (2)}}{=\!=} \lfloor u,a,t_i\rfloor = x_i$$

and so $\{x_1,\ldots,x_n\}\subset u^{\blacklozenge}$. This with (3) proves (1). Q.E.D.

(6.9) Theorem Let L be a canopy libra. Then $\overset{L}{\diamond}$ is an incidence relation.

Proof. Assume that $\overset{L}{\diamond}$ were not an incidence relation. Then there would exist a doubleton $\{a,b\}\subset L$ such that $b\in a^{\blacklozenge}$. Since L is a canopy libra, there would exist $c\in(L\cap b^{\blacklozenge})$. Since L is a canopy libra, there would exist $n\in\mathbb{N}$ odd and $\{x_1,\ldots,x_n\}\subset a^{\blacklozenge}$ such that $c=\lfloor x_1,\ldots,x_n\rfloor$. But then

$$\lfloor b,c,b\rfloor = \lfloor b,\lfloor x_1,\ldots,x_n\rfloor,b\rfloor = \lfloor b,x_n,\ldots,x_1,b\rfloor = \lfloor b,x_n,b,b,\ldots,b,b,_1,b\rfloor =$$

$$\lfloor\lfloor b,x_n,b\rfloor,\ldots \lfloor b,x_1,b\rfloor\rfloor = \lfloor x_1,\ldots,x_n\rfloor = c$$

which would be absurd. Q.E.D.

7. A Quadric Surface

(7.1) Purpose In this section we shall introduce an example which illustrates the concepts introduced in Sections (4), (5) and (6), and which provides motivation for much of the sequel.

(7.2) Real Projective Space Within that three-dimensional space E in which we seem to live, parallel lines seem to run together towards a point which we sometimes describe as lying "at infinity". We shall denote the set of all such "virtual" points by

$$\infty(\mathsf{E}) \,. \tag{1}$$

From each virtual point p emanates a maximal family of mutually parallel lines in E. We may change intuition to a logical system by defining

$$\mathbb{E} \equiv \mathsf{E} \cup \infty(\mathsf{E}) \,. \tag{2}$$

If L is a line in E, we shall write the virtual point towards which it tends as

$$\infty(\mathrm{L}) \tag{3}$$

There is a natural topology on $\mathbb{E}$ of which the restriction to E is the euclidean topology of E, and relative to which, E is an open subset of $\mathbb{E}$.[11] For any subset A of $\mathbb{E}$, we shall write

$$\overline{\mathrm{A}} \tag{4}$$

for the topological closure of A in $\mathbb{E}$. By a **line in** $\mathbb{E}$ we shall mean either $\overline{\mathrm{L}}$ for a line L in E, or $\overline{\mathrm{P}} \cap \infty(\mathsf{E})$ for a plane P in E. We shall say that $\infty(\mathsf{E})$ is the **plane at infinity** and that any line of $\mathbb{E}$ which lies entirely in the plane at infinity to be a **line at infinity**. By a **plane in** $\mathbb{E}$, we shall either mean the plane at infinity or $\overline{\mathrm{P}}$, for P a plane in E. The set $\mathbb{E}$ is called **real projective space**. We shall refer to the family of lines in $\mathbb{E}$ as

$$\mathbb{E}^{\boldsymbol{lines}} \tag{5}$$

and the family of planes in $\mathbb{E}$ as

$$\mathbb{E}^{\boldsymbol{planes}} \,. \tag{6}$$

By a **projective subset of** $\mathbb{E}$, we shall mean either a point, a line or a plane in $\mathbb{E}$. The smallest projective set containing a given set A as a subset, will be written

$$\overset{\vdash\!\dashv}{\mathrm{A}} \,. \tag{7}$$

If $\{a_1, \ldots, a_n\}$ is a finite subset of $\mathbb{E}$, we shall sometimes employ the notation

[11] To describe a basic open neighborhood of a point $\infty(\mathrm{L})$ of $\infty(\mathsf{E})$, we consider any elliptic hyperpoloid H of E such that the line L passes through both sheets. An elliptic hyperboloid H separates $\mathsf{E} \triangle \mathrm{H}$ into two components: one of these contains complete lines and the other does not — we shall call the **inside of** H the component which does not contain complete lines and denote it by H′. Let $\mathcal{F}$ denote the family of lines K which intersect both sheets of H. Then $\bigcup_{\mathrm{K} \in \mathcal{F}} \infty(\mathrm{K}) \cup (\mathrm{K} \cap \mathrm{H}')$ is a basic neighborhood of $\infty(\mathrm{L})$.

$$\overleftrightarrow{a_1,\ldots,a_n} \equiv \overline{\{a_1,\ldots,a_n\}} \,. \tag{8}$$

Thus, for two distinct points a and b of $\mathbb{E}$, $\overleftrightarrow{a,b}$ denotes the line in $\mathbb{E}$ determined by a and b.

The reason for adding virtual points to the space in which we live is the wonderful symmetry thereby achieved:

(1) every pair of distinct points determines a line – every pair of distinct planes determines a line;

(2) each disjoint point and line determines a plane – each line and each plane not containing that line determine a point;

(3) each intersecting pair of distinct lines determines both a point and a plane.

These simple observations (1) through (3) lie on the surface of a deeper and more curious symmetry induced by the so-called "polar mappings". A bijection $\phi\big|\mathbb{E}\hookrightarrow\mathbb{E}^{\mathbf{planes}}$ is said to be a **polar mapping** provided

$$(\forall\,\{x,y\}\subset\mathbb{E})\quad x\in\phi(y) \iff y\in\phi(x) \tag{9}$$

and that

$$\mathbf{Poles}(\phi) \equiv \{x\in\mathbb{E} : x\in\phi(x)\} \tag{10}$$

is non-void. The elements of the set $\mathbf{Poles}(\phi)$ are called the **poles of ϕ**.

By an **automorphism of $\mathbb{E}$** we shall mean a permutation of $\mathbb{E}$ which maps lines onto lines. The **conjugate of an automorphism θ** is the function

$$\theta^*\big|\mathbb{E}^{\mathbf{planes}}\ni P\hookrightarrow\{x\in\mathbb{E} : \theta(x)\in P\}\in\mathbb{E}^{\mathbf{planes}}\,. \tag{11}$$

We say that two subsets A and B of $\mathbb{E}$ are **qualitatively the same** provided that there exists an automorphism θ of $\mathbb{E}$ such that[12]

$$\overrightarrow{\theta}(A) = B \tag{12}$$

and that two polar mappings α and β are **qualitatively the same** provided that

$$\theta^*\circ\alpha\circ\theta = \beta\,. \tag{13}$$

It turns out that two polar mappings are qualitatively the same if, and only if, their sets of poles are qualitatively the same.

The set of poles of a polar mapping is called a **quadric surface**, or sometimes just a **quadric**. The condition of being qualitatively the same is an equivalence relation on the family of all quadrics, and relative to that equivalence relation there are exactly two classes. We shall call one class **spheroidal** and one class **toroidal**.

(7.3) Spheroidal Quadric Surfaces Spheroidal quadric surfaces are those of which their intersections with E are either ellipsoids, elliptic paraboloids or elliptic hyperboloids of two sheets. Spheroidal quadrics coming from ellipsoids of course do not intersect $\infty(\mathsf{E})$, spheroidal quadrics coming from elliptic paraboloids are tangent to $\infty(\mathsf{E})$ and spheroidal quadrics coming from elliptic hyperboloids of two sheets intersect $\infty(\mathsf{E})$ on a closed curve.

---

12 We recall that $\overrightarrow{\theta}(A)\equiv\{\theta(x) : x\in A\}$.

Let S be a spheroidal quadric surface. Let σ denote the polar mapping of which

$$S = \mathbf{Poles}(\sigma)\ . \qquad (1)$$

For any point $x \in S$, $\sigma(x)$ is the plane in $\mathbb{E}$ tangent to S at x. Obviously S separates its complement in $\mathbb{E}$ into two connected components, of which one contains complete lines: we call that component the **outside of** S — the other component is the **inside of** S.[13] If x is a point outside S, then there exists a cone with vertex S of which the lines are tangent to S — the points of tangency comprise the intersection of a plane $\sigma(x)$ with S. If x is a point inside S, then $\sigma(x)$ is the locus of all points y such that $x \in \sigma(y)$.

By a **spheroidal polar mapping** we shall mean a polar mapping of which the quadric surface of poles is a spheroidal quadric surface.

(7.4) Toroidal Quadric Surfaces Toroidal quadric surfaces are those of which their intersections with **E** are either hyperbolic paraboloids or elliptic hyperboloids of one sheet. Toroidal quadrics coming from hyperbolic paraboloids are tangent to $\infty(\mathbf{E})$, while those coming from elliptic hyperboloids intersect $\infty(\mathbf{E})$ at a closed curve. The complement of a toroidal quadric has two components, both of which contain complete lines, and these components are qualitatively the same. We shall call these disjoint components the **halves of real projective space relative to the toroidal quadric surface**.

Probably the simplest example is a circular hyperboloid T.[14]

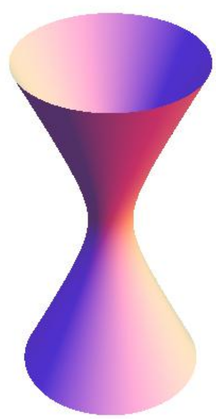


Fig. 9: Section of a Toroidal Quadric (from a circular hyperboloid)

As distinct from spheroidal quadrics, which contain no lines, toroidal quadrics are unions of lines. In fact, any three lines which are pairwise disjoint are contained in a unique toroidal quadric. For three such lines K, M and N, we shall show how such a quadric $\mathbf{Quadric}(K,M,N)$ is constructed.

Construction. Each element m of M is on exactly one plane containing K. The intersection of that plane with N is a singleton $n(m,K,M)$. Then

$$\bigcup_{x \in M} \overleftrightarrow{m, n(m,K,M)} = \mathbf{Quadric}(K,M,N)\ . \qquad (1)$$

[13] Evidently the concepts of outside and inside extend to general spheroidal quadric surfaces. The toroidal quadric surfaces do not have an outside and an inside.

[14] Sometimes called a hyperboloid of revolution, since it can be formed by rotating a hyperbola around the principal axis which does not intersect the hyperbola.

The family of lines in (1) is called a **regulus** and each such line is called a **rule of the regulus**. If we take any three distinct lines K´ , M´ and N´ from this regulus, then these lines are pairwise disjoint and

$$\mathbf{Quadric}(K,M,N) = \mathbf{Quadric}(K',M',N'.) \tag{2}$$

Furthermore, the regulus obtained from the analogous construction using K´ , M´ and N´ , is disjoint from the other regulus and the union of the two reguli comprises all the lines in the quadric. For the example of Figure (9), we illustrate these two reguli as follows:

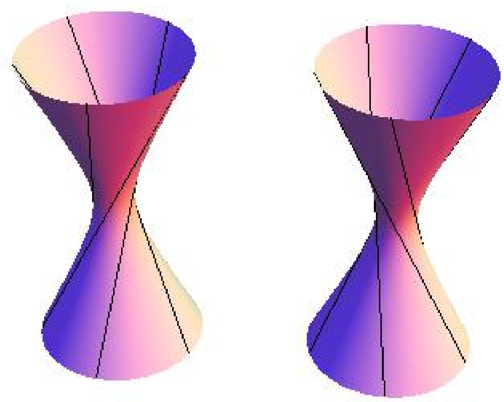


Fig. 10: Elements of the Two Reguli of a Toroidal Quadric

If we write $\boxed{T}$ and $\boxed{T}$ for the two reguli of a toroidal quadric surface T , then, for each $M \in \boxed{T}$ and $N \in \boxed{T}$, $M \cap N$ is a singleton in T — we shall denote this singleton by $M \wedge N$. It is evident from Figure (10) that there is a natural topology on each of the reguli such that each one of them is homeomorphic to the circle. The bijection

$$\boxed{T} \times \boxed{T} \ni [M,N] \hookrightarrow M \wedge N \in T \tag{3}$$

is actually a homeomorphism, and so T is homeomorphic to a torus.[15]

Let τ denote the polar mapping satisfying

$$T = \mathbf{Poles}(\tau) \,. \tag{4}$$

For $x \in T$, $\tau(x)$ is the plane tangent to T at x . This plane contains two lines lying in T: one $\tau_{\rceil}(x)$ a member of $\boxed{T}$ and one $\tau_{\lceil}(x)$ a member of $\boxed{T}$. Thus

$$(\forall x \in T) \quad \tau(x) \;=\; \overline{\tau_{\rceil}(x) \cup \tau_{\lceil}(x)} \,. \tag{5}$$

15 It is for this reason that we call such quadric surfaces toroidal.

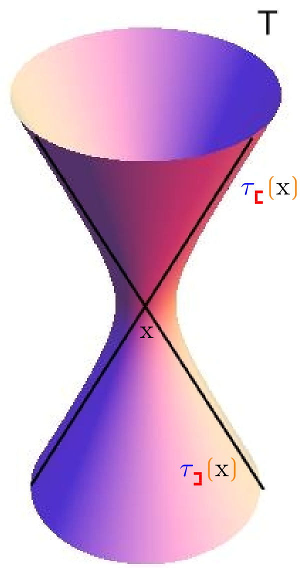


Fig. 11: Rules through a Point

For $x \in \mathbb{E} \triangle T$, $\tau(x)$ is the plane of $\mathbb{E}$ of which the intersection with T is the set of all points $t \in T$ such that $\overleftrightarrow{t,x}$ is tangent to T. We define

$$\mathbb{L} \equiv \mathbb{E} \triangle T. \tag{6}$$

Related to the restriction of τ to $\mathbb{L}$ is the mapping

$$\dot{\tau} \big| \mathbb{L} \ni x \hookrightarrow \dot{\tau}_x \in T^T \tag{7}$$

where,

$$(\forall [x,t] \in \mathbb{L} \times T) \quad \overleftrightarrow{x,t} \cap T = \{t, \dot{\tau}_x(t)\}. \tag{8}$$

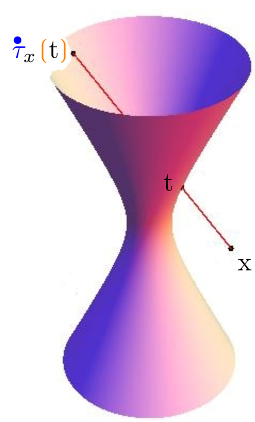


Fig. 12: Involutions of T

Thus, for each $x \in \mathbb{L}$ and $t \in T$, $\dot{\tau}_x(t)$ is the point other than t on the line $\overleftrightarrow{x,t}$. Of course $\dot{\tau}_x(t) = t$ if the line $\overleftrightarrow{x,t}$ is tangent to T. The connection between τ and $\dot{\tau}$ comes from the fact that, for each $x \in \mathbb{L}$ and $t \in T$,

$$\{(\tau_{\lrcorner}(x)\,(t))\wedge(\tau_{\llcorner}(x)\,(\dot{\tau}_x(t))),(\tau_{\llcorner}(x)\,(t))\wedge(\tau_{\lrcorner}(x)\,(\dot{\tau}_x(t)))\}\subset\tau(x)\,. \tag{9}$$

For each $x\in\mathbb{L}$, we shall write $\urcorner\tau\ulcorner_x$ to the restriction of $\overrightarrow{(\dot{\tau}_x)}$ to the regulus $\boxed{\underline{T}}$, thus,

$$(\forall\,[x,L]\in\mathbb{L}\times\boxed{\underline{T}})\quad \urcorner\tau\ulcorner_x(L)=\{\dot{\tau}_x(t):t\in L\}\,. \tag{10}$$

For $x\in\mathbb{L}$ and L in $\boxed{\underline{T}}$, the image $\urcorner\tau\ulcorner_x(L)$ is a line in $\boxed{\overline{T}}$ and is the intersection of T with the rest of the plane determined by x and L,

$$(\forall\,x\in\mathbb{L})\quad \urcorner\tau\ulcorner_x\,|\,\boxed{\underline{T}}\ni L\hookrightarrow\Big(\big(\overline{(\{x\}\cup L)}\,\triangle\,L\big)\cap T\Big)\in\boxed{\overline{T}}\,. \tag{11}$$

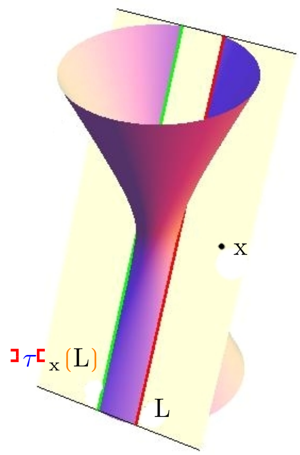


Fig. 13: The Function $\urcorner\tau\ulcorner$

It is a remarkable but true fact that the range $\ulcorner\urcorner\tau\ulcorner\urcorner$ of $\urcorner\tau\ulcorner$ is a balanced subset of the function libra $\boxed{\overline{T}}^{\boxed{\underline{T}}}!$. Since the function $\mathbb{L}\ni x\hookrightarrow\urcorner\tau\ulcorner_x\in\ulcorner\urcorner\tau\ulcorner\urcorner$ is a bijection, we can define a libra operator $\lfloor,,\rfloor$ on $\mathbb{L}$ to be the unique libra operator on $\mathbb{L}$ such that

$$(\forall\,[x,y,z]\in\mathbb{L}\times\mathbb{L}\times\mathbb{L})\quad \urcorner\tau\ulcorner_{\lfloor x,y,z\rfloor}\;=\;\llbracket\urcorner\tau\ulcorner_x,\urcorner\tau\ulcorner_y,\urcorner\tau\ulcorner_z\rrbracket\,. \tag{12}$$

In the construction of the preceding paragraph, the reguli $\boxed{\overline{T}}$ and $\boxed{\underline{T}}$ could have been interchanged to obtain a function $\ulcorner\tau\urcorner$ such that

$$(\forall\,x\in\mathbb{L})\quad \ulcorner\tau\urcorner_x\,|\,\boxed{\overline{T}}\ni L\hookrightarrow\Big(\big(\overline{(\{x\}\cup L)}\,\triangle\,L\big)\cap T\Big)\in\boxed{\underline{T}}\,. \tag{13}$$

The libra operator on $\mathbb{L}$ which would have in this way been induced, is just the obverse $\lceil,,\rceil$ of the libra operator $\lfloor,,\rfloor$.

By a **toroidal polar mapping** we shall mean a polar mapping of which the quadric surface of poles is a toroidal quadric surface.

(7.5) The Libra Operator on $\mathbb{L}$ For any subset A of $\mathbb{E}$, we define the **trace** $\widehat{A}$ **of** A to be just

$$\widehat{A}\equiv A\cap\mathbb{L}\,. \tag{1}$$

Thus we may speak of **line traces** and **plane traces**.

Let $\lfloor,,\rfloor$ be the libra operator on $\mathbb{L}$ as defined in (7.4.12). The balanced subsets of $\mathbb{L}$ are the singletons, the line traces and the plane traces $\widehat{P}$ for the planes P tangent to T.

We recall the libra twining relation of (6.2):

$$\overset{\mathbb{L}}{\diamond} \equiv \{[\mathrm{x,y}]\in\mathbb{L}\times\mathbb{L} : \lfloor \mathrm{x,y,x} \rfloor = \mathrm{y} \quad \text{and} \quad \mathrm{x}\neq\mathrm{y}\} \tag{2}$$

and the associated operators

$$(\forall\, \mathrm{x}\in\mathbb{L}) \quad \mathrm{x}^{\blacklozenge} = \{\mathrm{y}\in\mathbb{L}\,\triangle\,\{\mathrm{x}\} : \lfloor \mathrm{x,y,x} \rfloor = \mathrm{y}\} \tag{3}$$

and

$$(\forall\, \mathrm{A}\subset\mathbb{L}) \quad \mathrm{A}^{\bullet} = \bigcap_{\mathrm{a}\in\mathrm{A}} \mathrm{a}^{\blacklozenge}. \tag{4}$$

We recall that a subset A of $\mathbb{L}$ is said to be a span if

$$\mathrm{A} = \mathrm{A}^{\bullet\bullet}. \tag{5}$$

It is not difficult to check that

$$\{\mathrm{x}^{\blacklozenge} : \mathrm{x}\in\mathbb{L}\} = \{\widehat{\tau(\mathrm{x})} : \mathrm{x}\in\mathbb{L}\} \tag{6}$$

which is the family of plane traces $\widehat{\mathrm{P}}$, where P is a plane in $\mathbb{E}$ not tangent to T. It follows that, for any line trace $\widehat{\mathrm{K}}$ of a line $\mathrm{K}\not\subset\mathrm{T}$, $\widehat{\mathrm{K}}^{\bullet}$ is the intersection of the plane traces $\widehat{\tau(\mathrm{x})}$, for x in K. Thus

$$(\forall\, \mathrm{K}\in\mathbb{E}^{\mathbf{lines}}\,\triangle\,(\boxed{\mathrm{T}}\cup\boxed{\mathrm{T}})) \quad \widehat{\mathrm{K}}^{\bullet} \text{ is a line trace.} \tag{7}$$

Relative to T, we may separate the lines in $\mathbb{E}$ into four types: those which are subsets of T, those which intersect T in two points (**incident**), those which are tangent to T (**accident**) and those which are disjoint from T (**excident**). The following figure illustrates the situation for each of the last three:

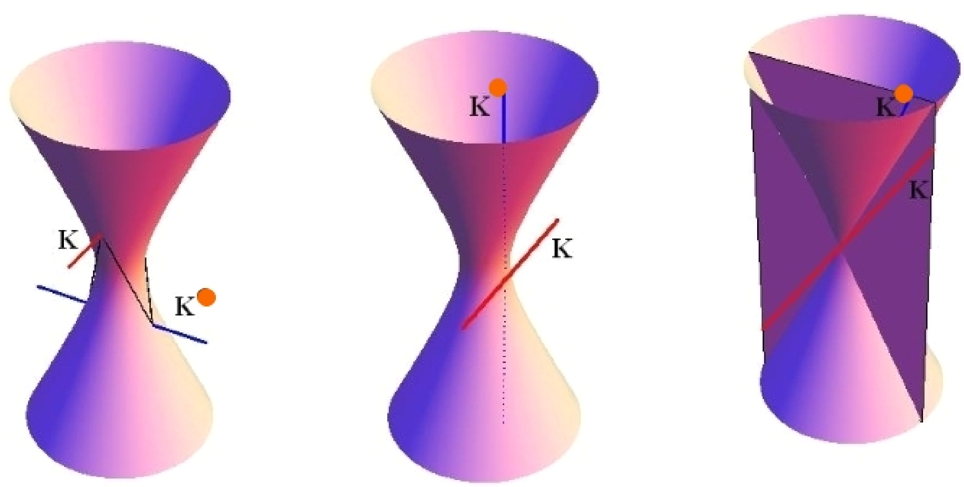


Fig. 14: Incident, Excident and Accident Line Traces and their Duals

We note that an accident line trace and its dual are on a common plane trace of a plane tangent to T.

Line traces, besides being spans, are balanced. It follows from Theorem (6.3) that line traces have an abelian libra structure. The corresponding libra groups are quite familiar: the groups corresponding to incidental line traces are all isomorphic to the multiplicative group of real numbers; the groups corresponding to excidental lines are all isomorphic to the group of complex numbers of unit modulus and the groups corresponding to accidental line traces are all isomorphic to the additive group of real numbers.

Let B be the trace of a plane tangent to T at a point q. For each element $b\in B$, it is evident that $\dot{\tau}_b(q) = q$ and that

$$(\forall x\in\mathbb{L}) \quad {}^{\urcorner}\tau^{\ulcorner}{}_x(\tau_{\llcorner}(q)) = \tau_{\lrcorner}(q) \iff x\in B\,. \tag{8}$$

It follows that B is balanced in $\mathbb{L}$. It is abelian as well. The set $B^{\bullet}$ is void, and so

$$B^{\bullet\bullet} = \mathbb{L} \tag{9}$$

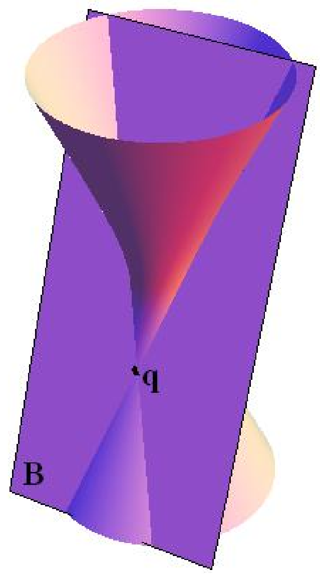


Fig. 15: A Plane B Tangent to T at a Point q Showing the Lines $\tau_{\lrcorner}(q)$ and $\tau_{\llcorner}(q)$

(7.6) Normal Simplices A simplex in $\mathbb{E}$ is a quadrupleton $S = \{p,q,r,s\}$ such that, for each $x\in S$, x is not in the smallest projective subset of $\mathbb{E}$ containing $S \triangle \{x\}$.

Let $\phi \big| \mathbb{E} \hookrightarrow \mathbb{E}^{planes}$ be any bijection such that (7.2.9) holds. If L is a line in $\mathbb{E}$ and P a plane in $\mathbb{E}$, then we shall say that L **and** P **are normal relative to** ϕ provided that $L \not\subset P$ and there exists $p\in L$ such that $\phi(p) = P$. If S is a simplex such that, for each $x\in S$, $(S \triangle \{x\}) \subset \phi(x)$, then we shall say that S is a ϕ-**normal simplex**.

Suppose that W is any plane in $\mathbb{E}$ and that $w \equiv \phi^{-1}(W)$. For any x in W, $\phi(x)$ intersects W in a line L. For any $y\in L$, $\phi(y)$ intersects L in a singleton $\{z\}$. Evidently $\{w,x,y,z\}$ is a ϕ-normal simplex. We have shown that

$$(\forall w\in\mathbb{E})(\exists \{x,y,z\} \subset \phi(w)) \quad \{w,x,y,z\} \text{ is a } \phi\text{-normal simplex.} \tag{1}$$

Now suppose that ϕ is a toroidal polar mapping τ, and that $\{w,x,y,z\}$ is a simplex, normal with respect to ϕ. We shall show:

$$w = \lfloor x,y,z \rfloor\,. \tag{2}$$

Proof. If x equaled $\lfloor x,y,z \rfloor$, then $\lfloor x,z,z \rfloor = \lfloor x,y,z \rfloor$ and so

$$y = \lfloor z,z,y,x,x \rfloor = \lfloor z, \lfloor x,y,z \rfloor, x \rfloor = \lfloor z, \lfloor x,z,z \rfloor, x \rfloor = \lfloor z,z,z,x,x \rfloor = z$$

which would be absurd. A similar argument shows that $\lfloor x,y,z \rfloor \neq z$. If $\lfloor x,y,z \rfloor$ equaled y , then, since

$$\lfloor x,y,z \rfloor = \lfloor x,y,x,x,z \rfloor = \lfloor y,x,z \rfloor ,$$

we would have $\lfloor y,x,x \rfloor = y = \lfloor y,x,z \rfloor$ and so

$$x = \lfloor x,y,y,x,x \rfloor = \lfloor x,y,\lfloor y,x,x \rfloor \rfloor = \lfloor x,y \lfloor y,x,z \rfloor \rfloor = \lfloor x,y,y,x,z \rfloor = z$$

which would be absurd. We have thus far shown that

$$\lfloor x,y,z \rfloor \notin \{x,y,z\} . \tag{3}$$

We have

$$\lfloor x,\lfloor x,y,z \rfloor,x \rfloor = \lfloor x,z,y,x,x \rfloor = \lfloor x,z,y,z,z \rfloor = \lfloor x,\lfloor z,y,z \rfloor,z \rfloor = \lfloor x,y,z \rfloor ,$$

$$\lfloor y,\lfloor x,y,z \rfloor,y \rfloor = \lfloor y,z,y,x,y \rfloor = \lfloor x,x,y,x,x,z,x,x,y,x,y \rfloor = \lfloor x,\lfloor x,y,x \rfloor,\lfloor x,z,x \rfloor,\lfloor x,y,x \rfloor,y \rfloor = \lfloor x,y,z,y,y \rfloor = \lfloor x,y,z \rfloor$$

and
$$\lfloor z,\lfloor x,y,z \rfloor,z \rfloor = \lfloor z,z,y,x,z \rfloor = \lfloor x,x,y,x,z \rfloor = \lfloor x,\lfloor x,y,x \rfloor,z \rfloor = \lfloor x,y,z \rfloor$$

which taken together, along with (3), implies that

$$\lfloor x,y,z \rfloor \in x^{\blacklozenge} \cap y^{\blacklozenge} \cap z^{\blacklozenge} \;=\; \tau(x) \cap \tau(y) \cap \tau(z) \;=\; \{w\} .$$

It follows that $\lfloor x,y,z \rfloor = w$. Q.E.D.

(7.7) Meridian Conic Sections Consider a plane in $\mathbb{E}$ which is not tangent to T . That plane is the image $\tau(p)$ for some point $p \in \mathbb{L}$. We shall write C for the intersection of that plane with T , and P for the trace of that plane. The union of the lines $\overleftrightarrow{p,c}$ for $c \in C$ is a cone K and C is a conic section of that cone. The set C has a meridian structure as given in (1.9) and is a circular meridian relative to this structure. The family of functions

$$\{C \ni c \hookrightarrow \dot{\tau}_x(c) \in C : x \in P\} \tag{1}$$

is the meridian family of involutions for C . The function $\gamma_{[} \,|\, C \ni c \hookrightarrow \tau_{[}(c) \in \sqsubset\!\mathrm{T}$ is a bijection and so induces a meridian structure on $\sqsubset\!\mathrm{T}$. Similarly the function $\gamma_{]} \,|\, C \ni c \hookrightarrow \tau_{]}(c) \in \mathrm{T}\!\sqsupset$ is a bijection and so induces a meridian structure on $\mathrm{T}\!\sqsupset$. Evidently $\gamma_{]} \circ (\gamma_{[})^{-1}$ is a meridian isomorphism from $\sqsubset\!\mathrm{T}$ onto $\mathrm{T}\!\sqsupset$. It can be described precisely as

$$\sqsubset\!\mathrm{T} \ni J \hookrightarrow \tau_{]}(J \wedge C) \in \mathrm{T}\!\sqsupset \text{ or, more succinctly, } {]}\tau{[}_{p} . \tag{2}$$

It is not difficult to show that all meridian isomorphisms from $\sqsubset\!\mathrm{T}$ onto $\mathrm{T}\!\sqsupset$ arise in this way: that

$$\{{]}\tau{[}_{x} : x \in \mathbb{L}\} \text{ is the family of meridian isomorphisms from } \sqsubset\!\mathrm{T} \text{ onto } \mathrm{T}\!\sqsupset . \tag{3}$$

One consequence of (3) is that

$$\{\phi_{C,x} \,|\, C \ni c \hookrightarrow \big(({]}\tau{[}_{x}(\tau_{[}(c)))\wedge C\big) \in C : x \in \mathbb{L}\} \text{ is the group of homographies of } C . \tag{4}$$

It is in fact possible to describe the various types of meridian homographies in terms of the geometry of $\mathbb{E}$. Recall that K is the cone with vertex p containing C . The cone K separates $\mathbb{E}$ into two components — we shall call R that component containing the inside of C and D the component containing the outside of C . The function $\phi_{C,p}$ is the identity of $\mathcal{G}$ and

$$\{\phi_{C,x} : x \in K \triangle \{p\}\} \quad \text{is the family of translations of C}, \tag{5}$$

$$\{\phi_{C,x} : x \in R\} \quad \text{is the family of pure rotations of C} \tag{6}$$

and

$$\{\phi_{C,x} : x \in D\} \quad \text{is the family of dilations of C}. \tag{7}$$

Now let x be any element of $\mathbb{L}$. Since $\phi_{C,x}$ is a homography of C, it follows from (1.3.5) that there exist involutions $\phi_{C,a}$ and $\phi_{C,b}$ of C such that

$$\phi_{C,x} = \phi_{C,a} \circ \phi_{C,b}\,. \tag{8}$$

That $\phi_{C,a}$ and $\phi_{C,b}$ are involutions means that $\{a,b\} \subset P$. It follows from (7) that

$$\lfloor a,p,b \rfloor = x\,. \tag{9}$$

Using the construction of (7.6) one can obtain a ϕ_p-normal simplex $\{p,b,r,s\} \subset \mathbb{L}$. We note that by that construction the points r and s must be in P. It follows from (7.6.2) that $b = \lfloor p,r,s \rfloor$. It now follows from (8) that

$$x = \lfloor a,p,\lfloor p,r,s \rfloor \rfloor = \lfloor a,r,s \rfloor\,. \tag{10}$$

Since $\{a,r,s\} \subset P$, we have shown that

$$\mathbb{L} \text{ is a canopy libra} \tag{11}$$

in the sense of (6.6).

(7.8) A Cartesian Aggregate of Balanced Sets Let τ be a toroidal polar mapping and let T be its set of poles. For each subset A of $\mathbb{E}$, we shall denote

$$A^{*} \equiv \{P \cap \mathbb{L} : P \in \mathbb{E}^{\mathbf{planes}} \text{ and } A \subset P\}\,. \tag{1}$$

It follows from (7.7.10) that the traces of planes which are not tangent to T cannot be balanced in $\mathbb{L}$. However planes tangent to T are balanced. In fact the set

$$\mathcal{T} \equiv \{\mathbb{L} \cap (\tau(t)) : t \in T\} \tag{2}$$

constitutes an aggregate of balanced sets. For any line J in $\boxed{T}$, the we have

$$J^{*} \in ||||(\mathcal{T}) \tag{3}$$

and in fact the function $\boxed{T} \ni J \hookrightarrow J^{*} \in ||||(\mathcal{T})$ is a bijection. Similarly

$$\boxed{T} \ni K \hookrightarrow K^{*} \in ≣(T) \quad \text{is a bijection.} \tag{4}$$

Condition (5.11.1) in the present context is that, given any doubleton $\{a,b\} \subset \mathbb{L}$, there is a plane P tangent to T such that $a \in P$ and $b \notin P$. Condition (5.11.2) is that, given any plane trace P in $\mathcal{T}$ and any set $\{a,b\} \subset \mathbb{L}$ such that $\lfloor a,P,b \rfloor = P$, both a and b must be in P. It can be shown that both these conditions are fulfilled, and so

$$\mathcal{T} \quad \text{is a cartesion aggregate of balanced sets.} \tag{5}$$

Let a and b be any two elements of $\mathbb{L}$ and P any element of $\mathcal{T}$. Then the plane $\overline{\mathrm{P}}$ is tangent to T at a point $\mathrm{p}: \tau(\mathrm{p}) = \overline{\mathrm{P}}$. Then, if p´ is any point in P, we have

$$\lfloor a,p',P \rfloor = \mathbb{L} \cap \overline{\{a\} \cup \tau_{\llcorner}(p)} : \tag{6}$$

the trace of the plane determined by a and the line in $\boxed{T}$ which lies in $\overline{\mathrm{P}}$. Similarly,

$$\lfloor P,p',a \rfloor = \mathbb{L} \cap \overline{\{a\} \cup \tau_{\lrcorner}(p)} : \tag{7}$$

the trace of the plane determined by a and the line in $\boxed{T}$ which lies in $\overline{\mathrm{P}}$. Furthermore

$$a \circledcirc b(P) = \lfloor a,P,b \rfloor = \mathbb{L} \cap \overline{(\tau_{\llcorner}(\dot{\tau}_a(p))) \cup (\tau_{\lrcorner}(\dot{\tau}_b(p)))} . \tag{8}$$

and, in particular,

$$a^{\circledcirc}(P) = \lfloor a,P,a \rfloor = \mathbb{L} \cap \tau(\dot{\tau}_a(p)) . \tag{9}$$

If J is any line in $\boxed{T}$, then

$$a \boxed{\mathcal{T}} b(J^*) = ({}^{\urcorner}\tau^{\ulcorner}{}_a(J))^* \tag{10}$$

and, if K is any line in $\boxed{T}$,

$$a \boxed{\mathcal{T}} b(K^*) = ({}^{\ulcorner}\tau^{\urcorner}{}_b(K))^* . \tag{11}$$

In particular

$$(\forall J \in \boxed{T}) \quad a^{\boxed{\mathcal{T}}}(J^*) = ({}^{\urcorner}\tau^{\ulcorner}{}_a(J))^* . \tag{12}$$

To more readily illustrate the structure of $\mathbb{L}$ induced by a diagonal, we suppose that T is generated by taking a circle C in euclidian space, taking a line L tangent to C lying in a vertical plane tangent to C at the point of tangency of L, and such that L makes an angle of $\pi/4$ with the plane determined by C . We then move the line continuously around the circle maintaining the angle and the property that the line continues to lie in a vertical tangent plane. The locus of the points on these lines is as indicated by the following figure. We write P for the plane determined by the circle C .[16]

[16] Alternatively we could construct an orthogonal rectangular coordinate system $[x,y,z]$, denote by C the set

$$\{t \in \mathbf{E} : z(t) = 0 \text{ and } x(t)^2 + y(t)^2 = 1\} .$$

and denote by T the quadric surface of which the intersection with $\mathbf{E}$ is

$$\{t \in \mathbf{E} : x(t)^2 + y(t)^2 = 1 + z(t)^2\} .$$

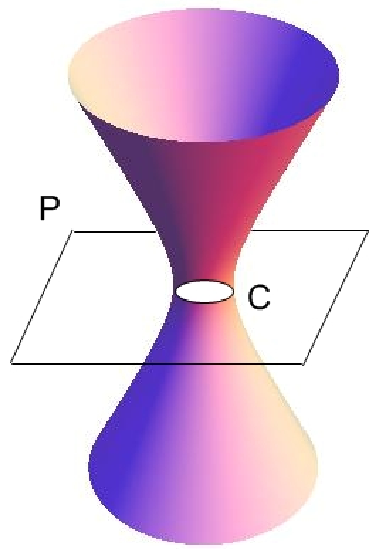


Fig. 16: Section of a Quadric from a Circle

Two plane traces in $\mathcal{T}$ are skew if their line of intersection is not contained in T.

Let p be the point in $\infty(\mathbf{E})$ such that each line in $\mathbb{E}$ through p is a vertical line. Then $\tau(\mathrm{p})$ is just P, which is also $\mathrm{p}^{\diamond}$. The diagonal $\backslash\mathrm{p}\backslash$ determined by p is then the family of all those vertical planes which are tangent to C, hence $\backslash\mathrm{p}\backslash$ is in one to one correspondence with the points of C.

Suppose that A is an element of $\mathcal{T}$ not contained in the diagonal $\backslash\mathrm{p}\backslash$. Then $\overline{\mathrm{A}}$ intersects C in two points a_1 and a_2. The plane trace symmetric to A with respect to p is just the trace of the other plane in $\mathcal{T}$ which passes through a_1 and a_2. Thus two plane traces A and B are p-skew if their line of intersection is not contained in T and is not contained in P (*i.e.* $\tau(\mathrm{p})$).

(7.9) Representations The function $\urcorner\tau\ulcorner$ is a cartesian representation of $\mathbb{L}$ from the set $\boxed{\mathrm{T}}$ to the set $\boxed{\mathrm{T}}$: in fact this representation was used to define the libra operator on $\mathbb{L}$. The function $\ulcorner\tau\urcorner$ is the obverse representation of $\urcorner\tau\ulcorner$.

The left $\mathcal{T}$-inner representation λ of $\mathbb{L}$ is of course equivalent to $\urcorner\tau\ulcorner$. The equivalence between the two is shown by the following equality:

$$(\forall\,[a,J]\in\mathbb{L}\times\boxed{\mathrm{T}})\quad (\urcorner\tau\ulcorner_{a}(J))^{*}=\lambda_{a}(J^{*})\,. \tag{1}$$

We now turn to a natural representation of the symmetrization of the libra operator $\lfloor,,\rfloor$ on $\mathbb{L}$. We consider the set $\mathbb{L}\times\mathbb{L}$ with the libra operator

$$(\forall\,[[x,y,z],[a,b,c]]\in\mathbb{L}\times\mathbb{L})\quad \lceil [x,a],[y,b],[z,c]\rceil=[\lfloor x,y,z\rfloor,\lceil a,b,c\rceil]=[\lfloor x,y,z\rfloor,\lfloor c,b,a\rfloor]\,. \tag{2}$$

We define $\delta\big|\mathbb{L}\times\mathbb{L}\hookrightarrow \mathrm{T}^{\mathrm{T}}$ by, for all $[a,b]\in\mathbb{L}\times\mathbb{L}$,

$$(\forall\,x\in\mathrm{T})\quad \delta_{[a,b]}(x)\equiv(\urcorner\tau\ulcorner_{a}(x))\wedge(\ulcorner\tau\urcorner_{b}(x))\,. \tag{3}$$

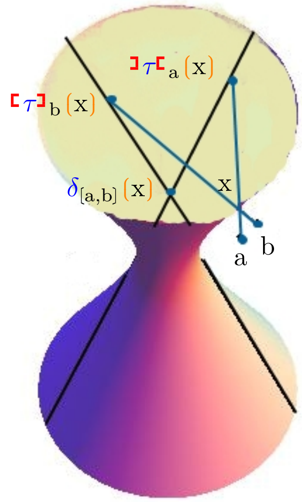


Fig. 17: Symmetrization Representation

This representation is equivalent to the symmetrization of the representation $\rfloor\tau\lfloor$, and so also equivalent to the symmetrization of the $\mathcal{T}$-inner representation λ of $\mathbb{L}$. As we know from (5.17.3), this latter representation is equivalent to the representation

$$\mathbb{L}\times\mathbb{L}\ni[x,a]\hookrightarrow x\,\textcircled{τ}\,a\in\mathcal{T}^{\mathcal{T}} \ . \tag{4}$$

The family $\{x^{\textcircled{$\tau$}}:x\in\mathbb{L}\}$ is a subfamily of the range $\{x\,\textcircled{$\tau$}\,a:\{x,a\}\subset\mathbb{L}\}$ of the above representation. For a general meridian, one may ask the question: is $\{x\,\textcircled{$\tau$}\,a:\{x,a\}\subset\mathbb{L}\}$ the smallest balanced subset of the function libra $\mathcal{T}^{\mathcal{T}}!$ containing $\{x^{\textcircled{$\tau$}}:x\in\mathbb{L}\}$? In the present context, that question is equivalent to whether or not

$$\{\delta_{[x,a]}:[x,a]\subset\mathbb{L}\}\text{ is the smallest balanced set containing }\{\delta_{[x,x]}:[x,x]\subset\mathbb{L}\}=\ulcorner(\dot{\tau})\urcorner\ ? \tag{5}$$

We shall defer the answer to (9.7) *infra*.

8. Meridian Libras

(8.1) Discussion The group of homographies of a meridian is a function libra $\mathfrak{L}$. In this section we characterize which libras appear in this way and investigate their properties.

(8.2) Notation Let $\mathcal{T}$ be a homogeneous aggregate of balanced subsets of a libra L. For $\mathcal{A}\subset\mathcal{T}$ and $\{a,b\}\subset L$ we define

$$\lfloor a,\mathcal{A},b\rfloor \equiv \{\lfloor a,A,b\rfloor : A\in\mathcal{A}\}. \tag{1}$$

(8.3) Theorem Let $\mathcal{T}$ be a cartesian aggregate of balanced subsets of a libra L such that $\mathcal{T}$ has dimension at least 4. Suppose that there exists $a\in L$ such that

$$(\forall\, B\in\mathcal{T}) \quad B\cap a^{\blacklozenge} \text{ is balanced and has more than one element} \tag{1}$$

and

$$(\forall\,\{B,D\}\subset\mathcal{T} : D \text{ and } B \text{ are a-skew}) \quad B\cap D\cap a^{\blacklozenge} \text{ is a singleton.} \tag{2}$$

Let

$$\mathcal{M}_a(\Vert\Vert(\mathcal{T})) \equiv \{(a^{\boxed{\mathcal{T}}})^{-1}\circ x^{\boxed{\mathcal{T}}} : x\in a^{\blacklozenge}\}. \tag{3}$$

Then $\mathcal{M}_a(\Vert\Vert(\mathcal{T}))$ is a meridian family of involutions on $\Vert\Vert(\mathcal{T})$.

Proof. For $\{x,a\}\subset L$ such that $\lfloor x,a,x\rfloor = a$

$$((a^{\boxed{\mathcal{T}}})^{-1}\circ x^{\boxed{\mathcal{T}}})\circ((a^{\boxed{\mathcal{T}}})^{-1}\circ x^{\boxed{\mathcal{T}}}) = (a^{\boxed{\mathcal{T}}})^{-1}\circ(x^{\boxed{\mathcal{T}}}\circ(a^{\boxed{\mathcal{T}}})^{-1}\circ x^{\boxed{\mathcal{T}}}) \overset{\text{by (5.6)}}{=\!=} (a^{\boxed{\mathcal{T}}})^{-1}\circ(\lfloor x,a,x\rfloor)^{\boxed{\mathcal{T}}} = (a^{\boxed{\mathcal{T}}})^{-1}\circ a^{\boxed{\mathcal{T}}}$$

which shows that $(a^{\boxed{\mathcal{T}}})^{-1}\circ x^{\boxed{\mathcal{T}}}$ is an involution.

Suppose that $\{\mathcal{A},\mathcal{B},\mathcal{D},\mathcal{E}\}\subset \Vert\Vert(T)$ satisfies $\{\mathcal{A},\mathcal{E}\}\cap\{\mathcal{B},\mathcal{D}\}=\emptyset$. If

$$\lfloor a,\mathcal{A}\wedge a^{\boxed{\mathcal{T}}}(\mathcal{E}),a\rfloor = \mathcal{B}\wedge a^{\boxed{\mathcal{T}}}(\mathcal{D}),$$

then (5.15) would imply that $\mathcal{E}=\mathcal{D}$, which would be absurd. It follows that $\mathcal{A}\wedge a^{\boxed{\mathcal{T}}}(\mathcal{E})$ and $\mathcal{B}\wedge a^{\boxed{\mathcal{T}}}(\mathcal{D})$ are distinct and skew, and so (2) implies that

$$(\exists\, j\in L) \quad (\mathcal{A}\wedge a^{\boxed{\mathcal{T}}}(\mathcal{E}))\cap(\mathcal{B}\wedge a^{\boxed{\mathcal{T}}}(\mathcal{D}))\cap a^{\blacklozenge} = \{j\}.$$

Thus $(a^{\boxed{\mathcal{T}}})^{-1}\circ j^{\boxed{\mathcal{T}}}$ sends $\mathcal{A}$ to $\mathcal{E}$ and $\mathcal{B}$ to $\mathcal{D}$. If any other $\phi\in\mathcal{M}_a(\Vert\Vert(\mathcal{T}))$ did the same, since

$$\phi = (a^{\boxed{\mathcal{T}}})^{-1}\circ k^{\boxed{\mathcal{T}}},$$

we would have k in

$$(\mathcal{A}\wedge a^{\boxed{\mathcal{T}}}(\mathcal{E}))\cap(\mathcal{B}\wedge a^{\boxed{\mathcal{T}}}(\mathcal{D}))\cap a^{\blacklozenge},$$

whence $k=j$ and $\phi=(a^{\boxed{\mathcal{T}}})^{-1}\circ j^{\boxed{\mathcal{T}}}$. This establishes (1.2.4).

Let $\mathcal{P}$ and $\mathcal{Q}$ be in $\Vert\Vert(T)$. Suppose that $\{\beta,\gamma,\delta\}\subset\mathcal{M}_a(\Vert\Vert(\mathcal{T}))$ is such that $\beta(\mathcal{P})=\gamma(\mathcal{P})=\delta(\mathcal{P})=\mathcal{Q}$. By (3) there exists $\{b,c,d\}\subset a^{\blacklozenge}$ such that

$$(a^{\boxed{\mathcal{T}}})^{-1}\circ b^{\boxed{\mathcal{T}}}=\beta,\ (a^{\boxed{\mathcal{T}}})^{-1}\circ c^{\boxed{\mathcal{T}}}=\gamma \text{ and } (a^{\boxed{\mathcal{T}}})^{-1}\circ d^{\boxed{\mathcal{T}}}=\delta.$$

Let P be the element of $\mathcal{P}$ which contains a and let Q be in $\mathcal{Q}$. Let $R\equiv \boxed{Q}\wedge\boxed{P}$. Then

$$\boxed{Q} = \mathcal{Q} = \beta(\mathcal{P}) = (a^{\boxed{\mathcal{T}}})^{-1}\circ b^{\boxed{\mathcal{T}}}(\boxed{P}) = (a^{\boxed{\mathcal{T}}})^{-1}(\boxed{\lfloor b,P,b\rfloor}) = \boxed{\lfloor a,\lfloor b,P,b\rfloor,a\rfloor} =$$

$$\boxed{\lfloor a,b,\lfloor P,b,a\rfloor\rfloor} = \boxed{\lfloor P,b,a\rfloor} = \boxed{\lfloor P,\lfloor b,a,b\rfloor,b\rfloor} = \boxed{\lfloor P,a,b\rfloor}$$

which implies that

$$\lfloor P,a,b\rfloor \in \boxed{Q} \cap \boxed{P} \implies \lfloor P,a,b\rfloor = \boxed{Q} \wedge \boxed{P} = R\,.$$

Thus $b = \lfloor a,a,b\rfloor \in \lfloor P,a,b\rfloor = R$. Similarly, c and d are in R as well. Hence $\{b,c,d\} \subset R \cap a^{\blacklozenge}$. In view of (1) this implies that $\lfloor b,c,d\rfloor$ is in $R \cap a^{\blacklozenge}$. From (6.3) it follows that

$$\lfloor b,c,d\rfloor = \lfloor d,c,b\rfloor\,. \tag{4}$$

Now

$$((a^{\boxed{T}})^{-1}\circ b^{\boxed{T}})\circ((a^{\boxed{T}})^{-1}\circ c^{\boxed{T}})\circ((a^{\boxed{T}})^{-1}\circ d^{\boxed{T}})(\mathcal{P}) = ((a^{\boxed{T}})^{-1}\circ b^{\boxed{T}})\circ((a^{\boxed{T}})^{-1}\circ c^{\boxed{T}})\circ((a^{\boxed{T}})^{-1}\circ d^{\boxed{T}})(\boxed{P}) =$$

$$\boxed{\lfloor a,\lfloor b,\lfloor a,\lfloor c,\lfloor a,\lfloor d,P,d\rfloor,a\rfloor,c\rfloor,a\rfloor,b\rfloor,a\rfloor} = \boxed{\lfloor\lfloor\lfloor a,b,a\rfloor,c,\lfloor a,d,a\rfloor,a,P,a,\lfloor a,d,a\rfloor,c,\lfloor a,b,a\rfloor\rfloor\rfloor} =$$

$$\boxed{\lfloor b,c,d,a,P,a,d,c,b\rfloor} = \boxed{\lfloor a,\lfloor a\lfloor b,c,d\rfloor,a\rfloor,P,\lfloor a\lfloor d,c,b\rfloor,a\rfloor,a\rfloor} \overset{\text{by (4)}}{=\!=\!=} \boxed{\lfloor a,\lfloor b,c,d\rfloor,P,\lfloor b,c,d\rfloor,a\rfloor} =$$

$$(a^{\boxed{T}})^{-1}\circ\lfloor a,b,c\rfloor^{\boxed{T}}(\boxed{P}) = (a^{\boxed{T}})^{-1}\circ\lfloor a,b,c\rfloor^{\boxed{T}}(\mathcal{P})$$

which establishes (1.2.5).

Let β and γ be in $\mathcal{M}_a(\Vert\Vert(\mathcal{T}))$. Choose $\{b,c\} \subset a^{\blacklozenge}$ such that $\beta = (a^{\boxed{T}})^{-1}\circ b^{\boxed{T}}$ and $\gamma = (a^{\boxed{T}})^{-1}\circ c^{\boxed{T}}$. Then

$$\beta\circ\gamma^{-1}\circ\beta = (a^{\boxed{T}})^{-1}\circ b^{\boxed{T}}\circ(c^{\boxed{T}})^{-1}\circ a^{\boxed{T}}\circ(a^{\boxed{T}})^{-1}\circ b^{\boxed{T}} = (a^{\boxed{T}})^{-1}\circ b^{\boxed{T}}\circ(c^{\boxed{T}})^{-1}\circ b^{\boxed{T}} = (a^{\boxed{T}})^{-1}\circ\lfloor b,c,b\rfloor^{\boxed{T}}\,. \tag{5}$$

Furthermore

$$\lfloor a,\lfloor b,c,b\rfloor,a\rfloor = \lfloor a,b,a,\lfloor a,c,a\rfloor,\lfloor a.b,a\rfloor\rfloor = \lfloor b,c,b\rfloor$$

which implies that $\lfloor b,c,b\rfloor \in a^{\blacklozenge}$. This, with (5), establishes (1.2.6). Q.E.D.

(8.4) Remarks Consider the toroid function libra $\mathbb{L}$ of Section (7). The B of (8.3.1) is the trace of a plane tangent to T and the $a^{\blacklozenge}$ of (8.3.1) is the trace of some plane not tangent to T. Thus, in this case, (8.3.1) states that the intersection of two such plane traces has more than one element. In reality, it is a line trace.

The B and D of (8.3.2) are the traces of two planes tangent to T such that $a^{\textcircled{T}}(B) \neq D$. It follows that the line trace $B\cap D$ is not contained in the plane trace $a^{\blacklozenge}$, and so the intersection of the two is indeed a point.

(8.5) Definition and Remarks A cartesian aggregate of balanced sets satisfying (8.3.1) and (8.3.2), and of dimension at least 4, will be called a **meridian aggregate**. Thus $\Vert\Vert(\mathcal{T})$ is a meridian with meridian family of involutions $\mathcal{M}_a(\Vert\Vert(\mathcal{T}))$. We define

$$\mathcal{G}_a(\Vert\Vert(\mathcal{T})) \tag{1}$$

to be the group of homographies of this meridian.

Evidently $\equiv(\mathcal{T})$ is a meridian as well, relative to the meridian family of involutions

$$\{a^{\boxed{T}}\circ(x^{\boxed{T}})^{-1} : x \in a^{\blacklozenge}\}\,. \tag{2}$$

We define

$$\mathcal{M}_a(\equiv(\mathcal{T})) \quad \text{and} \quad \mathcal{G}_a(\equiv(\mathcal{T})) \tag{3}$$

analogously.

(8.6) Remarks Consider the toroid function libra $\mathbb{L}$, the quadric T, the element $a \in \mathbb{L}$ and the circle C of (7.7.7). The mapping $C \ni t \hookrightarrow ({}^{\lrcorner}\tau^{\llcorner}{}_t)^* \in ||||(\mathcal{T})$ is a meridian isomorphism.

(8.7) Theorem Let $\mathcal{T}$ be a meridian aggregate of balanced subsets of a libra L. Then, for all $b \in L$

$$(\forall B \in \mathcal{T}) \quad B \cap b^{\blacklozenge} \text{ is balanced and has more than one element} \tag{1}$$

and

$$(\forall \{B,C\} \subset \mathcal{T} \text{ b-skew}) \quad B \cap C \cap b^{\blacklozenge} \text{ is a singleton.} \tag{2}$$

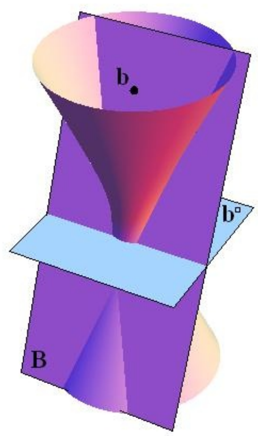


Fig. 18: A Tangent Plane and a Span Plane

If a and $\mathcal{M}_a(||||(\mathcal{T}))$ are as in (8.3.1), then

$$\mathcal{M}_a(||||(\mathcal{T})) = \{(b^{\boxed{\mathcal{T}}})^{-1} \circ x^{\boxed{\mathcal{T}}} : x \in b^{\blacklozenge}\}. \tag{3}$$

Furthermore, if L is a properly linear libra, then

$$\mathcal{G}_b(||||(\mathcal{T})) = \{(b^{\boxed{\mathcal{T}}})^{-1} \circ x^{\boxed{\mathcal{T}}} : x \in L\}. \tag{4}$$

Proof. $[\overset{(1)}{\Longrightarrow}]$ Let B be in $\mathcal{T}$. Let a be as in (8.3) and let b be in L. From (8.3.1) we know that there exists a doubleton $\{r,s\} \in \lfloor a,b,B \rfloor \cap a^{\blacklozenge}$: let $x \equiv \lfloor b,a,r \rfloor$ and $y \equiv \lfloor b,a,s \rfloor$. Evidently $\{x,y\} \subset B$ and $x \neq y$. We have

$$\lfloor b,x,b \rfloor = \lfloor b,b,a,r,b \rfloor = \lfloor \lfloor a,r,a \rfloor,a,b \rfloor = \lfloor a,a,b \rfloor = b \tag{5}$$

and so x is in $b^{\blacklozenge}$.Similarly, y is in $b^{\blacklozenge}$ as well. We have shown that $B \cap b^{\blacklozenge}$ has more than one element.

Let x, y and z be any elements of $B \cap b^{\blacklozenge}$. Evidently $\{\lfloor a,b,x \rfloor, \lfloor a,b,y \rfloor, \lfloor a,b,z \rfloor\}$ is a subset of $\lfloor a,b,B \rfloor \cap x^{\blacklozenge}$ and so (8.3.1) implies that there exists $u \in \lfloor a,b,B \rfloor$ such that $u = \lfloor \lfloor a,b,x \rfloor, \lfloor a,b,y \rfloor, \lfloor a,b,z \rfloor \rfloor$. Let $w \in B$ be such that $u = \lfloor a,b,w \rfloor$. We have

$$\lfloor x,y,z\rfloor = \lfloor b,a,a,b,x,y,b,a,a,b,z\rfloor = \lfloor b,a,\lfloor a,b,x\rfloor,\lfloor a,b,y\rfloor,\lfloor a,b,z\rfloor\rfloor = \lfloor b,a,u\rfloor = w$$

which shows that $B\cap b^{\blacklozenge}$ is balanced. It follows that (1) holds.

$[\overset{(2)}{\Longrightarrow}]$ Let now B and C be in $\mathcal{T}$ and be b-skew. Then $\lfloor a,b,B\rfloor$ and $\lfloor a,b,C\rfloor$ are $\lfloor a,b,b\rfloor$-skew: *i.e.* they are a-skew. By (8.3.2), $\lfloor a,b,B\rfloor\cap\lfloor a,b,C\rfloor\cap a^{\blacklozenge}$ is a singleton. Hence the intersection of $B=\lfloor b,a,\lfloor a,b,B\rfloor\rfloor$, $C=\lfloor b,a,\lfloor a,b,C\rfloor\rfloor$ and $b^{\blacklozenge}=\lfloor b,a,a\rfloor^{\blacklozenge}$ is a singleton. This establishes (2).

$[\overset{(3)}{\Longrightarrow}]$ Let ϕ be any element of $\mathcal{M}_a$. There there exists $y\in a^{\blacklozenge}$ such that $\phi=(a^{\boxed{\mathcal{T}}})^{-1}\circ y^{\boxed{\mathcal{T}}}$. Let $x\equiv\lfloor b,a,y\rfloor$. As above we see that x is in $b^{\blacklozenge}$ and, since

$$\phi=(b^{\boxed{\mathcal{T}}})^{-1}\circ(b^{\boxed{\mathcal{T}}}\circ(a^{\boxed{\mathcal{T}}})^{-1}\circ y^{\boxed{\mathcal{T}}})=(b^{\boxed{\mathcal{T}}})^{-1}\circ x^{\boxed{\mathcal{T}}},$$

it follows that $\mathcal{M}_a(\Vert\Vert(\mathcal{T}))\subset\{(b^{\boxed{\mathcal{T}}})^{-1}\circ x^{\boxed{\mathcal{T}}}:x\in b^{\blacklozenge}\}$. The reverse inclusion follows by a similar argument.

$[\overset{(4)}{\Longrightarrow}]$ Since $\mathcal{M}_b$ is a meridian, it follows from (1.3.5) that

$$\mathcal{G}_b\subset\{(b^{\boxed{\mathcal{T}}})^{-1}\circ x^{\boxed{\mathcal{T}}}:x\in L\} \tag{6}$$

Suppose now that L is a properly linear libra and that x is any element of L. Let $\{\mathcal{A},\mathcal{B},\mathcal{C}\}$ be any tripleton in $\Vert\Vert(T)$. By (1.2.2), there exists $\phi\in\mathcal{M}_b$ such that

$$\phi(\mathcal{A})=(b^{\boxed{\mathcal{T}}})^{-1}\circ x^{\boxed{\mathcal{T}}}(\mathcal{A}),\quad \phi(\mathcal{B})=(b^{\boxed{\mathcal{T}}})^{-1}\circ x^{\boxed{\mathcal{T}}}(\mathcal{B})\quad\text{and}\quad \phi(\mathcal{C})=(b^{\boxed{\mathcal{T}}})^{-1}\circ x^{\boxed{\mathcal{T}}}(\mathcal{C}). \tag{7}$$

By (1.3.5), there exists $\{y,z\}\in b^{\blacklozenge}$ such that $\big((b^{\boxed{\mathcal{T}}})^{-1}\circ y^{\boxed{\mathcal{T}}}\big)\circ\big((b^{\boxed{\mathcal{T}}})^{-1}\circ z^{\boxed{\mathcal{T}}}\big)=\phi$. Since L is a proper line libra, there exists $p\in L$ such that $\{x,\lfloor y,b,z\rfloor\}\subset p^{\blacklozenge}$ From (7) follows

$$(b^{\boxed{\mathcal{T}}})^{-1}\circ\lfloor y,b,z\rfloor^{\boxed{\mathcal{T}}}(\mathcal{A})=(b^{\boxed{\mathcal{T}}})^{-1}\circ\lfloor y,b,z\rfloor^{\boxed{\mathcal{T}}}(\mathcal{A}),\quad (b^{\boxed{\mathcal{T}}})^{-1}\circ\lfloor y,b,z\rfloor^{\boxed{\mathcal{T}}}(\mathcal{B})=(b^{\boxed{\mathcal{T}}})^{-1}\circ x^{\boxed{\mathcal{T}}}(\mathcal{B})$$

and

$$(b^{\boxed{\mathcal{T}}})^{-1}\circ\lfloor y,b,z\rfloor^{\boxed{\mathcal{T}}}(\mathcal{C})=(b^{\boxed{\mathcal{T}}})^{-1}\circ x^{\boxed{\mathcal{T}}}(\mathcal{C})$$

whence follows that

$$(p^{\boxed{\mathcal{T}}})^{-1}\circ\lfloor y,b,z\rfloor^{\boxed{\mathcal{T}}}(\mathcal{A})=(b^{\boxed{\mathcal{T}}})^{-1}\circ\lfloor y,b,z\rfloor^{\boxed{\mathcal{T}}}(\mathcal{A}),\quad (p^{\boxed{\mathcal{T}}})^{-1}\circ\lfloor y,b,z\rfloor^{\boxed{\mathcal{T}}}(\mathcal{B})=(p^{\boxed{\mathcal{T}}})^{-1}\circ x^{\boxed{\mathcal{T}}}(\mathcal{B})$$

and

$$(p^{\boxed{\mathcal{T}}})^{-1}\circ\lfloor y,b,z\rfloor^{\boxed{\mathcal{T}}}(\mathcal{C})=(p^{\boxed{\mathcal{T}}})^{-1}\circ x^{\boxed{\mathcal{T}}}(\mathcal{C}).$$

From (1.2.2) follows that $(p^{\boxed{\mathcal{T}}})^{-1}\circ\lfloor y,b,z\rfloor^{\boxed{\mathcal{T}}}=(p^{\boxed{\mathcal{T}}})^{-1}\circ x^{\boxed{\mathcal{T}}}$. Consequently

$$(b^{\boxed{\mathcal{T}}})^{-1}\circ x^{\boxed{\mathcal{T}}}=(b^{\boxed{\mathcal{T}}})^{-1}\circ\lfloor y,b,z\rfloor^{\boxed{\mathcal{T}}}=\big((b^{\boxed{\mathcal{T}}})^{-1}\circ y^{\boxed{\mathcal{T}}}\big)\circ\big((b^{\boxed{\mathcal{T}}})^{-1}\circ z^{\boxed{\mathcal{T}}}\big)$$

which implies that x is in $\mathcal{G}_b$. Q.E.D.

(8.8) Notation and Definition It is a consequence of Theorem (8.7) that the sets $\mathcal{M}_a(\Vert\Vert(\mathcal{T}))$ are independent of a. Consequently we shall denote this family of involutions of $\Vert\Vert(\mathcal{T})$ simply by

$$\mathcal{M}(\Vert\Vert(\mathcal{T})). \tag{1}$$

and the function libra $\mathcal{G}_a$ by

$$\mathcal{G}(\Vert\Vert(\mathcal{T})). \tag{2}$$

For each $p\in L$, we know from (1.3.5) that

$$\mathcal{G}(\,||||(\mathcal{T}\,)\,) = \{\big((p^{\boxed{\mathcal{T}}})^{-1}\circ r^{\boxed{\mathcal{T}}}\big)\circ\big((p^{\boxed{\mathcal{T}}})^{-1}\circ s^{\boxed{\mathcal{T}}}\big) : \{r,x\}\subset p^{\blacklozenge}\}$$

whence follows that

$$(\forall\, p\in L) \quad \{\lfloor r,p,s\rfloor : \{r,s\}\subset p^{\blacklozenge}\} \text{ is a libra isomorphic with the libra } \mathcal{G}\,. \tag{3}$$

We know from (8.7.4) that

$$L = \{\lfloor r,p,s\rfloor : \{r,s\}\subset p^{\blacklozenge}\} \text{ for each } p\in L \text{ if } L \text{ is a properly linear libra } L: \tag{4}$$

whether the hypothesis that L be properly linear be a necessary condition to ensure this set equality, is not known to the author at this time.

We shall say that L is a **meridian libra** if it has a meridian aggregate and the equality of (4) holds.

(8.9) Theorem Let M be a meridian with meridian family $\mathcal{M}$ of involutions and group $\mathcal{G}$ group of homographies. Let L be the function libra $\mathcal{G}$ with libra operation

$$(\forall\, \alpha,\beta,\gamma) \quad [\![\alpha,\beta,\gamma]\!] = \alpha\circ\beta^{-1}\circ\gamma\,.$$

Let ι denote the identity mapping on the set $\mathcal{G}$, which also may be regarded as a representation of L from the set M to the set M. Let $\mathcal{T}$ be abbreviation for the aggregate $\mathcal{T}_\iota$ as defined in (4.4.5). Then

$$\iota \text{ is equivalent to the left inner representation of } L\,, \tag{1}$$

$$\mathcal{T} \text{ is a meridian aggregate} \tag{2}$$

and

$$L \text{ is a properly linear libra.} \tag{3}$$

Thus L is a meridian libra with meridian aggregate $\mathcal{T}$.

Proof. [(1)] Define the bijections

$$\mu\big|\, ||||(\mathcal{T}\,) \ni \boxed{m \overset{\iota}{=} n} \hookrightarrow n\in M \quad \text{and} \quad \nu\big|\, ||||(\mathcal{T}\,) \ni \boxed{m \overset{\iota}{=} n} \hookrightarrow m\in M\,.$$

Then, for $\alpha\in\mathcal{G}$ and $m\in M$,

$$\nu\circ\alpha^{\boxed{\mathcal{T}}}\circ\mu^{-1}(m) = \nu\circ\alpha^{\boxed{\mathcal{T}}}\big(\boxed{m \overset{\iota}{=} n}\big) = \nu\big(\boxed{\alpha(m) \overset{\iota}{=} \alpha^{-1}(m)}\big) = \alpha(m) = \iota_\alpha(m)$$

which establishes the equivalence.

[(2)] We first show that $\mathcal{T}$ is a cartesian aggregate, which, by the definition of this term, follows once we have shown that ι is a cartesian representation. Let then {x,r} and {y,s} be doubleton subsets of M. By (1.2.1) there exists $\phi\in\mathcal{G}$ such that $\phi(x) = y$ and $\phi(r) = s$. By (5.8.1) it follows that ι is a cartesian representation.

Next we show that (8.3.1) and (8.3.2) hold. Let then ι be the element a of L of (8.3.1) and (8.3.2). Then the element B of (8.3.1) is a family of the form $\{\phi\in\mathcal{G} : \phi(u) = v\}$, where {u,v} is a subset of M. The set $\iota^{\blacklozenge}$ is just $\mathcal{M}$. Since M has at least four elements, there exists a doubleton $\{r,s\}\subset M\bigtriangleup\{u,v\}$. By (1.2.4), there exists $\{\alpha,\beta\}\subset\mathcal{M}$ such that

$$\alpha(u) = v\,,\ \alpha(r) = r\,,\ \beta(u) = v \text{ and } \beta(r) = s\,.$$

Thus α and β are distinct elements of B. Than B is balanced follows from (1.2.5). This establishes (8.3.1).

Now let D be an element of $\mathcal{T}$ such that D and B are ι-skew. Then there exists $\{m,n\}\subset M$ such that $D = \{\phi\in\mathcal{G} : \phi(m) = n\}$. That B and D are ι-skew just means that $[u,v] \neq [m,n]$. It follows from (1.2.4) that there exists a unique element $\psi \in \mathcal{M}$ such that $\psi(\mathrm{u}) = \mathrm{v}$ and $\psi(\mathrm{m}) = \mathrm{n}$. This establishes (8.3.2).

[(3)] Let θ be any element of L. We shall first show that

$$\theta^{\blacklozenge} \cap \mathcal{M} \neq \emptyset . \tag{4}$$

In view of the Jordon canonical form, we may choose a tripleton $\{0,1,\infty\}$ of elements of M relative to which either

$$(\exists\, \mathrm{a} \in \mathrm{M}) \quad \theta = \begin{pmatrix} \mathrm{a} & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

or

$$(\exists\, \mathrm{a} \in \mathrm{M}) \quad \theta = \begin{pmatrix} \mathrm{a} & 1 \\ 0 & \mathrm{a} \end{pmatrix} \tag{6}$$

where notation is as in (1.6).

If (5) holds, we define

$$\psi \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and check that

$$\begin{pmatrix} \mathrm{a} & 0 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ \begin{pmatrix} \mathrm{a} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{a} \\ \mathrm{a} & 0 \end{pmatrix} = \mathrm{a} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which means that ψ is in $\theta^{\blacklozenge} \cap \mathcal{M}$. If (6) holds, we define

$$\psi \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and check that

$$\begin{pmatrix} \mathrm{a} & 1 \\ 0 & \mathrm{a} \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \circ \begin{pmatrix} \mathrm{a} & 1 \\ 0 & \mathrm{a} \end{pmatrix} = \begin{pmatrix} \mathrm{a}^2 & 0 \\ 0 & -\mathrm{a}^2 \end{pmatrix} = \mathrm{a}^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which means that ψ is in $\theta^{\blacklozenge} \cap \mathcal{M}$. We have established (5).

Now suppose that $\{\alpha,\beta\} \subset \mathrm{L}$. From (5) follows that there exists some χ in $(\alpha^{-1} \circ \beta)^{\blacklozenge} \cap \mathcal{M}$. Letting $\gamma \equiv \alpha \circ \chi$, we compute

$$[\![\gamma,\alpha,\gamma]\!] = [\![\alpha \circ \chi,\alpha,\alpha \circ \chi]\!] = \alpha \circ \chi \circ \alpha^{-1} \circ \alpha \circ \chi = \alpha \circ \chi \circ \chi = \alpha$$

which implies that γ is in $\alpha^{\blacklozenge}$. Furthermore

$$[\![\gamma,\beta,\gamma]\!] = [\![\alpha \circ \chi,\beta,\alpha \circ \chi]\!] = \alpha \circ \chi \circ \beta^{-1} \circ \alpha \circ \chi = \alpha \circ [\![\chi,\alpha^{-1} \circ \beta,\chi]\!] = \alpha \circ \alpha^{-1} \circ \beta = \beta .$$

Thus γ is in both $\alpha^{\blacklozenge}$ and $\beta^{\blacklozenge}$, which proves that L is a properly linear libra. Q.E.D.

(8.10) Definition, Notation and Discussion Let L be a meridian libra with a meridian aggregate $\mathcal{T}$ of balanced subsets of L. It being somewhat cumbersome to deal with $\mathcal{M}(\Vert\Vert(\mathcal{T}))$ and $\mathcal{G}(\Vert\Vert(\mathcal{T}))$, we shall at times deal with a representation ρ of L equivalent to the left $\mathcal{T}$-inner representation. Such a representation will be said to be **characteristic**. We shall reserve $\mu| \Vert\Vert(\mathcal{T}) \hookrightarrow \boxed{\rho}$ and $\nu| \equiv(\mathcal{T}) \hookrightarrow \boxed{\rho}$ to represent the bijections such that

$$(\forall\, \mathrm{a} \in \mathrm{L}) \quad \rho_{\mathrm{a}} = \nu \circ \mathrm{a}^{\boxed{\mathcal{T}}} \circ \mu^{-1} . \tag{1}$$

In this case we shall write M for $\lbrack\rho$ and N for $\rho\rbrack$. We shall write $\mathcal{G}$ for the family of homographies of the meridian M and $\mathcal{M}$ for the family of meridian involutions of M.

For a given $a\in L$ it is sometimes expedient to form the representation

$$\overset{\boxed{a}}{\rho} \equiv {\rho_a}^{-1}\circ\rho \tag{2}$$

of L which has the property that $M = \lbrack\overset{\boxed{a}}{\rho} = \overset{\boxed{a}}{\rho}\rbrack$. We shall say that $\overset{\boxed{a}}{\rho}$ is the a-**representation founded on** ρ. In this case we shall sometimes denote one element of M by ∞, write F for $M\triangle\{\infty\}$, and then choose two distinct elements 0 and 1 in F. We shall use the field operations of addition and multiplication defined in F as in (1.6). Such a choice of 0, 1 and ∞ will be called a **choice of a basis for** M. Adopting the terminology of (1.6.2), for $\{a,b,c,d\}\subset F$ and $x\in M$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x) \equiv \begin{cases} \dfrac{a\cdot x+b}{c\cdot x+d} & \text{if } x\in F \text{ and } c\cdot x+d\neq 0; \\ \dfrac{b}{d} & \text{if } x=\infty \text{ and } d\neq 0; \\ \infty & \text{otherwise.} \end{cases} \tag{3}$$

From Theorem (8.7) we know that M is a meridian where

$$\{\overset{\boxed{a}}{\rho}_x : x\in a^{\blacklozenge}\} = \mathcal{M}, \text{ the family of meridian involutions of } M \tag{4}$$

and

$$\{\overset{\boxed{a}}{\rho}_x : x\in L\} = \mathcal{G}, \text{ the family of homographies of } M. \tag{5}$$

(8.11) Remarks In the case where $L=\mathbb{L}$ as in (7.5), we take for ρ the function

$$\rbrack\tau\lbrack \,\big|\, \mathbb{L} \hookrightarrow \lbrack T^{\,T\rbrack}. \tag{1}$$

Thus in this case

$$M = \lbrack\rho = \lbrack T \quad\text{and}\quad N = \rho\rbrack = T\rbrack. \tag{2}$$

If $C\equiv a^{\blacklozenge}\cap T$, and

$$\gamma\,\big|\,\lbrack T \ni K \hookrightarrow K\wedge C\in C, \tag{3}$$

then

$$\mathbb{L}\ni a\hookrightarrow \gamma\circ a\circ\gamma^{-1}\in C^{C} \tag{4}$$

is a characteristic representation of $\mathbb{L}$ of which the range is the set of homographies of the circle meridian.

(8.12) Theorem Let L be a meridian libra with meridian aggregate $\mathcal{T}$. Let ρ be a characteristic representation of L, and let $\overset{\boxed{a}}{\rho}$ be a corresponding a-representation. Then

$$(\forall\,\{a,b,c\}\subset M \text{ and } \{u,v,w\}\in M \text{ tripletons})(\exists!\,x\in L)\quad \overset{\boxed{a}}{\rho}_x(a)=u,\ \overset{\boxed{a}}{\rho}_x(b)=v \text{ and } \overset{\boxed{a}}{\rho}_x(c)=w, \tag{1}$$

$$(\forall\,\{a,b,c\}\subset M \text{ and } \{r,s,t\}\in N \text{ tripletons})(\exists!\,x\in L)\quad \rho_x(a)=r,\ \rho_x(s)=v \text{ and } \rho_x(c)=t, \tag{2}$$

$$(\forall\, \{\mathcal{A},\mathcal{B},\mathcal{C}\} \subset \text{||||}(\mathcal{T}) \text{ and } \{\mathcal{U},\mathcal{V},\mathcal{W}\} \subset \equiv(\mathcal{T}) \text{ tripletons})(\exists!\, x \in L)$$

$$x^{\boxed{\mathcal{T}}}(\mathcal{A}) = \mathcal{U},\ x^{\boxed{\mathcal{T}}}(\mathcal{B}) = \mathcal{V} \text{ and } x^{\boxed{\mathcal{T}}}(\mathcal{C}) = \mathcal{W} \tag{3}$$

and

$$(\forall\, \{A,B,C\} \subset \mathcal{T} \text{ pairwise skew}) \quad A \cap B \cap C \text{ is a singleton.} \tag{4}$$

Proof. [$\overset{(1)}{\Longrightarrow}$] Follows from (1.2.1) applied to (8.7).

[$\overset{(2)}{\Longrightarrow}$] Let $\lfloor u,v,w \rfloor \equiv \lfloor r,s,t \rfloor$ and apply (1).

[$\overset{(3)}{\Longrightarrow}$] Follows from the fact that ρ and the left inner representation are equivalent.

[$\overset{(4)}{\Longrightarrow}$] Let A, B and C in $\mathcal{T}$ be pairwise skew. Then $\boxed{A}$, $\boxed{B}$ and $\boxed{C}$ are pairwise distinct, as are $\boxed{A}$, $\boxed{B}$ and $\boxed{C}$. By (3) there is a unique $x \in L$ such that

$$x^{\boxed{\mathcal{T}}}(\boxed{A}) = \boxed{A},\ x^{\boxed{\mathcal{T}}}(\boxed{B}) = \boxed{B} \text{ and } x^{\boxed{\mathcal{T}}}(\boxed{C}) = \boxed{C}.$$

By Theorem (5.18.2), this implies that $x \in A \cap B \cap C$. For any $y \in A \cap B \cap C$ it would also hold that

$$y^{\boxed{\mathcal{T}}}(\boxed{A}) = \boxed{A},\ y^{\boxed{\mathcal{T}}}(\boxed{B}) = \boxed{B} \text{ and } y^{\boxed{\mathcal{T}}}(\boxed{C}) = \boxed{C},$$

whence follows that y must be x. Q.E.D.

(8.13) Lemma Let L be a meridian libra with meridian aggregate $\mathcal{T}$. Let $\overset{\boxed{a}}{\rho}$ be an a-representation for $a \in L$. Let $b \in L$ be distinct from a. Then there exists a choice of basis and $\{q,r\} \subset F$ such that, if

$$A \equiv \{\begin{pmatrix} e & rd \\ qd & e \end{pmatrix} : \{e,d\} \subset F \text{ and } e^2 \neq qrd^2\} \text{ and } B \equiv \{\begin{pmatrix} e & -rd \\ qd & -e \end{pmatrix} : \{e,d\} \subset F \text{ and } e^2 \neq qrd^2\},$$

then

$$(\forall\, \{u,v\} \subset A \text{ a doubleton}) \quad \{u,v\}^{\bullet} = B, \tag{1}$$

$$(\forall\, \{g,h\} \subset B \text{ a doubleton}) \quad \{g,h\}^{\bullet} = A \tag{2}$$

and

$$A^{\bullet} = B \quad \text{and} \quad B^{\bullet} = A. \tag{3}$$

Furthermore, these choices can be made such that

$$\overset{\boxed{a}}{\rho}_b = \begin{pmatrix} 1 & r \\ q & 1 \end{pmatrix} \text{ if } b \notin a^{\bullet} \quad \text{and} \quad \overset{\boxed{a}}{\rho}_b = \begin{pmatrix} 0 & r \\ q & 0 \end{pmatrix} \text{ if } b \in a^{\bullet} \tag{4}$$

Proof. Suppose first that $b \in a^{\bullet}$. Let 0 be any element of M and set $\infty \equiv \rho_b(0)$. Let 1 be any third element distinct from 0 and ∞ and define $q \equiv 1$ and $r \equiv \overset{\boxed{a}}{\rho}_b(1)$.

[$\overset{(4)}{\Longrightarrow}$] Now suppose that $b \notin a^{\bullet}$. It follows from (1.3.5) that there exists $\pi \in \mathcal{M}$ with fixed points and $\sigma \in \mathcal{M}$ such that $\overset{\boxed{a}}{\rho}_b = \phi \circ \sigma$. Let 0 and ∞ be the fixed points of π and let 1 be any third element of M distinct from 0 and ∞. Then $\pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Since σ is self-inverse, there exist $\{i,j,k\} \subset F$ such that $\sigma = \begin{pmatrix} i & j \\ k & -i \end{pmatrix}$. Then

$$\overset{\boxed{a}}{\rho}_b = \pi \circ \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \circ \begin{pmatrix} i & j \\ k & -i \end{pmatrix} = \begin{pmatrix} i & j \\ -k & i \end{pmatrix}.$$

The hypothesis that $b \notin a^{\bullet}$ insures that i cannot by 0. We let $q \equiv \frac{-k}{i}$ and $r \equiv \frac{j}{i}$. It is now evident that (4) holds.

$[\overset{(1)}{\Longrightarrow}]$ Whenever $\{u,v\}\subset A$, direct calculation shows that

$$B\subset\{u,v\}^{\bullet}\,. \tag{5}$$

Let $\{u,v\}\subset A$ be a doubleton and choose $\{d,e,s,t\}\subset F$ of cardinality 4 such that

$$\overset{\boxed{a}}{\rho}_u=\begin{pmatrix} e & rd \\ qd & e \end{pmatrix} \quad \text{and} \quad e^2\neq qrd^2 \tag{6}$$

and

$$\overset{\boxed{a}}{\rho}_u=\begin{pmatrix} t & rs \\ qs & t \end{pmatrix} \quad \text{and} \quad t^2\neq qrs^2\,. \tag{7}$$

Let p be a member of $\{u,v\}^{\bullet}$ and choose $\{w,x,y,z\}\subset M$ such that

$$\overset{\boxed{a}}{\rho}_p=\begin{pmatrix} w & x \\ y & z \end{pmatrix} \quad \text{and} \quad wz\neq xy\,. \tag{8}$$

Then

$$\lfloor u,p,u\rfloor=p \iff \begin{pmatrix} e & rd \\ qd & e \end{pmatrix}\circ\begin{pmatrix} w & x \\ y & z \end{pmatrix}^{-1}=\begin{pmatrix} w & x \\ y & z \end{pmatrix}\circ\begin{pmatrix} e & rd \\ qd & e \end{pmatrix}^{-1} \iff$$
$$\begin{pmatrix} ez-rdy & rdw-ex \\ qdz-ey & ew-qdx \end{pmatrix}=\begin{pmatrix} ew-qdx & -rdw+ex \\ -qdz+ey & ez-rdy \end{pmatrix}. \tag{9}$$

Equality (9) implies that the matrix $\begin{bmatrix} ez-rdy & rdw-ex \\ qdz-dy & ew-qdx \end{bmatrix}$ is a non-0 multiple k of the matrix $\begin{bmatrix} ew-qdx & -(rdw-ex) \\ -(qdz-dy) & ez-rdy \end{bmatrix}$. If $k=-1$, then

$$rdw-rdy=-ew+qdx \tag{10}$$

and, if $k\neq-1$, then

$$rdw-ex=0=qdz-ey \quad \text{and} \quad ez-rdy=ew-qdx\,. \tag{11}$$

If $e=0$, then (11) and (6) imply that $w=0=z$ and $ry=qx$: thus

$$\overset{\boxed{a}}{\rho}_p=\begin{pmatrix} w & x \\ y & z \end{pmatrix}=\begin{pmatrix} 0 & \frac{ry}{q} \\ y & 0 \end{pmatrix}=\begin{pmatrix} 0 & ry \\ qy & 0 \end{pmatrix}=\begin{pmatrix} e & rd \\ qd & e \end{pmatrix}=\overset{\boxed{a}}{\rho}_u\,. \tag{12}$$

If $e\neq 0$, then (11) and (6) imply

$$x=r\frac{d}{e}w\,,\ y=q\frac{d}{e}z\,,\ z=r\frac{d}{e}y+w-q\frac{d}{e}x \Longrightarrow z=rq\,(\tfrac{d}{e})^2z+w-qr(\tfrac{d}{e})^2w \Longrightarrow$$
$$(1-\frac{d^2}{e^2}rq)\,z=(1-\frac{d^2}{e^2}qr)\,w \overset{\text{by (6)}}{=\!=\!=} z=w \Longrightarrow x=rdy \Longrightarrow$$
$$\overset{\boxed{a}}{\rho}_p=\begin{pmatrix} w & r\,(\frac{d}{e})\,w \\ q\,(\frac{d}{e})\,w & w \end{pmatrix}=\begin{pmatrix} e & rd \\ qd & e \end{pmatrix}=\overset{\boxed{a}}{\rho}_u\,. \tag{13}$$

Since u is not in $\{u,v\}^{\bullet}$ and p is, it follows from (12) and (13) that (11) cannot hold. It follows that

(10) must hold, and so

$$e(z+w)=d(qx+ry) \tag{14}$$

holds. An analogous argument, using v instead of u, yields

$$t(z+w)=s(qx+ry)\,. \tag{15}$$

From (14) and (15) follows that either

$$z+w=0=qx+ry \tag{16}$$

or

$$\frac{e}{t}=\frac{d}{s}\,. \tag{17}$$

If (17) held, then

$$\overset{\boxed{a}}{\rho}_u=\begin{pmatrix} e & rd \\ qd & e \end{pmatrix}=\begin{pmatrix} \frac{dt}{s} & rd \\ qd & \frac{dt}{s} \end{pmatrix}=\begin{pmatrix} t & rs \\ qs & t \end{pmatrix}=\overset{\boxed{a}}{\rho}_v$$

which would be absurd since $\overset{\boxed{a}}{\rho}$ is faithful. It follows that (16) holds and so

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix}=\begin{pmatrix} w & -\frac{ry}{q} \\ y & -w \end{pmatrix}=\begin{pmatrix} qw & -ry \\ qy & -qw \end{pmatrix},$$

which in turn implies that p is in B. thus $\{u,v\}^{\bullet}\subset B$. This, with (5), establishes (1).

$[\overset{(2)}{\Longrightarrow}]$ Whenever $\{g,h\}\subset B$, direct calculations shows that

$$A\subset\{g,h\}^{\bullet}\,. \tag{18}$$

Let $\{g,h\}\subset B$ be a doubleton and choose $\{d,e,s,t\}\subset F$ of cardinality 4 such that

$$\begin{pmatrix} e & -rd \\ qd & -e \end{pmatrix} \quad \text{and} \quad -e^2\neq qrd^2\,, \tag{19}$$

$$\begin{pmatrix} d & -rd \\ qs & -t \end{pmatrix} \quad \text{and} \quad -t^2\neq qrs^2\,. \tag{20}$$

Let p be a member of $\{g,h\}^{\bullet}$ and choose $\{w,x,y,z\}\subset M$ such that

$$\overset{\boxed{a}}{\rho}_p=\begin{pmatrix} w & x \\ y & z \end{pmatrix} \quad \text{and} \quad wz\neq xy\,. \tag{21}$$

Then

$$\lfloor g,p,g\rfloor=p \iff \begin{pmatrix} e & -rd \\ qd & -e \end{pmatrix}\circ\begin{pmatrix} w & x \\ y & z \end{pmatrix}^{-1}=\begin{pmatrix} w & x \\ y & z \end{pmatrix}\circ\begin{pmatrix} e & -rd \\ qd & -e \end{pmatrix}^{-1} \iff$$

$$\begin{pmatrix} ez+rdy & -ex-rdw \\ qdz+ey & -qdx-ew \end{pmatrix}=\begin{pmatrix} qdx+ew & -ex-rdw \\ qdz+ey & -ez-rdy \end{pmatrix}. \tag{22}$$

Equality (22) implies that the matrix $\begin{bmatrix} ez+rdy & -ex-rdw \\ qdz+ey & -qdx-ew \end{bmatrix}$ is a non-0 multiple k of the matrix

$\begin{bmatrix} qdx+ew & -ex-rdw \\ qdz+ey & -ez-rdy \end{bmatrix}$. If $k=1$, then

$$ez+rdy=qdx+ew \tag{23}$$

and if $k\neq 1$, then

$$-ez-rdw=0=qdz+ey \quad \text{and} \quad ez+rdy=-qdx-ew\,. \tag{24}$$

If $e=0$, then (24) and (19) imply that $w=0=z$ and $ry=-qx$: thus

$$\overset{\boxed{a}}{\rho}_p=\begin{pmatrix} w & x \\ y & z \end{pmatrix}=\begin{pmatrix} 0 & -\frac{ry}{q} \\ y & 0 \end{pmatrix}=\begin{pmatrix} e & -rd \\ qd & -e \end{pmatrix}=\overset{\boxed{a}}{\rho}_g\,. \tag{25}$$

If $e\neq 0$, then (24) and (19) imply that

$$x=-r(\tfrac{d}{e})w\,,\ y=-q(\tfrac{d}{e})z\,,\ z=(\tfrac{d}{e})y-w-q(\tfrac{d}{e})x\Longrightarrow z=rq(\tfrac{d}{e})^2z-w+qr(\tfrac{d}{e})^2w\Longrightarrow$$

$$(1-(\tfrac{d}{e})^2rq)z=-(1-(\tfrac{d}{e})^2rq)w\xRightarrow{\text{by (19)}}z=-w\Longrightarrow x=-rdy\Longrightarrow$$

$$\overset{\boxed{a}}{\rho}_p=\begin{pmatrix} w & -r(\frac{d}{e})w \\ q(\frac{d}{e})w & -w \end{pmatrix}=\begin{pmatrix} e & -rd \\ qd & -e \end{pmatrix}=\overset{\boxed{a}}{\rho}_g\,. \tag{26}$$

Since u is not in $\{u,v\}^{\bullet}$ and p is, it follows from (25) and (26) that (24) cannot hold. Thus (23) must hold, and so

$$e(z-w)=d(qx-ry) \tag{27}$$

holds. An analogous argument, using h instead of g, yields

$$t(z-w)=s(qx-ry)\,. \tag{28}$$

From (27) and (28) follows that either

$$z-w=0=qx-ry \tag{29}$$

or

$$\frac{e}{t}=\frac{d}{s}\,. \tag{30}$$

If (30) held, then

$$\overset{\boxed{a}}{\rho}_g=\begin{pmatrix} e & -rd \\ qd & -e \end{pmatrix}=\begin{pmatrix} \frac{dt}{s} & -rd \\ qd & -\frac{dt}{s} \end{pmatrix}=\begin{pmatrix} t & -rs \\ qs & -t \end{pmatrix}=\overset{\boxed{a}}{\rho}_h$$

which would be absurd since $\overset{\boxed{a}}{\rho}$ is faithful. It follows that (29) holds and so

$$\overset{\boxed{a}}{\rho}_p=\begin{pmatrix} w & x \\ y & z \end{pmatrix}=\begin{pmatrix} w & \frac{ry}{q} \\ y & w \end{pmatrix}=\begin{pmatrix} qw & ry \\ qy & qw \end{pmatrix},$$

which in turn implies that p is in B. Thus $\{g,h\}^{\bullet}\subset A$. This, with (18), establishes (2).

$[\xRightarrow{Y3}]$ That $A\subset A^{\bullet}$ follows from direct computation. Let g be in $A^{\bullet}$. Since and b are in A, it follows that g is in $\{a,b\}^{\bullet}$, which by (1) is just B. Hence $A^{\bullet}=\infty$. That $B^{\bullet}=\mathcal{T}$ can be shown analogously. Q.E.D.

(8.14) Theorem Let L be a meridian libra with meridian aggregate $\mathcal{T}$. Let a and b be distinct elements of L. Then

$$\{a,b\}^{\bullet} \text{ has at least three elements,} \tag{1}$$

$$\{a,b\}^{\bullet} \text{ is balanced,} \tag{2}$$

$$(\forall\,\{x,y\}\subset\{a,b\}^{\bullet} \text{ a doubleton})\quad \{a,b\}^{\bullet}=\{x,y\}^{\bullet\bullet} \tag{3}$$

and

$$(\forall\,\{c,d\}\subset\{a,b\}^{\bullet\bullet} \text{ a doubleton})\quad \{c,d\}^{\bullet}=\{a,b\}^{\bullet}. \tag{4}$$

Proof. $[\overset{(1)}{\Longrightarrow}]$ Let 0, 1, ∞, A, B, q and r be as in Lemma (8.13). Then both $\overset{\boxed{a}}{\rho}_a=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ and $\overset{\boxed{a}}{\rho}_b$ are in A, and are distinct. It follows from (8.13.1) that

$$\{a,b\}^{\bullet}=B. \tag{5}$$

We may and shall presume that $-1\neq qr\neq 1$. It follows that $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $\begin{pmatrix}1&-r\\q&1\end{pmatrix}$ and $\begin{pmatrix}1&r\\-q&1\end{pmatrix}$ are pairwise distinct elements of B. This, with (5), establishes (1).

$[\overset{(2)}{\Longrightarrow}]$ Follows from (5) and direct computation.

$[\overset{(3)}{\Longrightarrow}]$ Let x and y be distinct elements of $\{a,b\}^{\bullet}$. Then

$$\{a,b\}^{\bullet}\ \overset{\text{by (8.13.1)}}{=\!=\!=}\ B \Longrightarrow \{a,b\}^{\bullet\bullet}=\mathbb{B}^{\bullet}\ \overset{\text{by (8.13.3)}}{=\!=\!=}\ A\ \overset{\text{by (8.13.2)}}{=\!=\!=}\ \{x,y\}^{\bullet}$$

which is (3).

$[\overset{(4)}{\Longrightarrow}]$ In (3) we now replace x and y by c and d then replace a and b by x and y: this yields $\{c,d\}^{\bullet}=\{x,y\}^{\bullet\bullet}$. Hence $\{c,d\}^{\bullet}=\{a,b\}^{\bullet}$, which is (4). Q.E.D.

(8.15) Definition and Notation For distinct a and b in a meridian libra L we shall denote the set $\{a,b\}^{\bullet\bullet}$ by $\overleftrightarrow{a,b}$ and refer to it as a **line trace**. We adopt the notation

$$L^{\mathbf{lines}}\equiv\{\overleftrightarrow{a,b}:\{a,b\}\subset L \text{ is a doubleton}\}. \tag{1}$$

(8.16) Remark For the libra $\mathbb{L}$ of (8.1), the concept of a line trace introduced in (8.15) agrees with that already defined for $\mathbb{L}$ in (7.5).

(8.17) Theorem Let $\mathcal{T}$ be a meridian aggregate of a libra L and let K be an element of $L^{\mathbf{lines}}$. Then

$$K^{\bullet} \text{ is in } L^{\mathbf{lines}}, \tag{1}$$

$$K^{\bullet\bullet}=K, \tag{2}$$

$$K\cap K^{\bullet}=\emptyset, \tag{3}$$

$$K\cup K^{\bullet} \text{ is balanced} \tag{4}$$

and

$$(\forall\,\{x,y\}\subset K \text{ a doubleton})\quad \overleftrightarrow{x,y}=K. \tag{5}$$

Proof. $[\overset{(1)}{\Longrightarrow}]$ Let $\{a,b\}\subset L$ be a doubleton such that $\{a,b\}^{\bullet\bullet}=K$. By (8.13.1) and (8.13.3) there exist distinct x and y in $\{a,b\}^{\bullet}$ such that $\{a,b\}^{\bullet\bullet}=\{x,y\}^{\bullet}$. Thus

$$\overleftrightarrow{x,y}=\{x,y\}^{\bullet\bullet}=\{a,b\}^{\bullet\bullet\bullet}=K^{\bullet}$$

which establishes (1).

$[\overset{(2)}{\Longrightarrow}]$ That (2) holds follows from (6.1.5).

$[\overset{(3)}{\Longrightarrow}]$ If a were an element of $K\cap K^{\bullet}$, there would be another element b of K and we could apply (8.13) to obtain disjoint line traces $A=K$ and $B=K^{\bullet\bullet}$, which would be absurd.

$[\overset{(4)}{\Longrightarrow}]$ That (4) holds follows from (6.3).

$[\overset{(5)}{\Longrightarrow}]$ Let x and y be distinct elements of K. Then

$$K\overset{\text{by (2)}}{=\!=\!=}K^{\bullet\bullet}\overset{\text{by (8.13.4)}}{\Longrightarrow}\{x,y\}^{\bullet}=K^{\bullet}\Longrightarrow\overleftrightarrow{x,y}=\{x,y\}^{\bullet\bullet}=K^{\bullet\bullet}\overset{\text{by (8.13.2)}}{=\!=\!=}K,$$

which establishes (5). Q.E.D.

(8.18) Theorem Let L be a meridian libra with meridian aggregate $\mathcal{T}$. Let a be an element of L and B an element of $\mathcal{T}$. Then

$$(B\cap a^{\blacklozenge})\in L^{\mathbf{lines}} \tag{1}$$

and

$$a\in B\Longleftrightarrow(B\cap a^{\blacklozenge})^{\bullet}\subset B. \tag{2}$$

Proof. There are two cases to consider: either $B=[0\overset{\overset{\boxed{a}}{\rho}}{=\!=}0]$ for some $0\in M$, or $B=[0\overset{\overset{\boxed{a}}{\rho}}{=\!=}\infty]$ for distinct 0 and ∞ in M.

[Case I: $B=[0\overset{\overset{\boxed{a}}{\rho}}{=\!=}0]$ for some $0\in M$] Choose 1 and ∞ in M distinct from each other and from 0, and let $F\equiv M\vartriangle\{\infty\}$. Since $\overset{\boxed{a}}{\rho}_a=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, it follows that

$$\{\overset{\boxed{a}}{\rho}_x:x\in(B\cap a^{\blacklozenge})\}=\{\begin{pmatrix}1&0\\r&-1\end{pmatrix}:r\in F\}. \tag{3}$$

Let $\{u,v\}\subset(B\cap a^{\blacklozenge})$ satisgy $\overset{\boxed{a}}{\rho}_u=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and $\overset{\boxed{a}}{\rho}_v=\begin{pmatrix}1&0\\1&-1\end{pmatrix}$. For $\{w,x,y,z\}\subset F$ we have

$$[\![\overset{\boxed{a}}{\rho}_u,\begin{pmatrix}w&x\\y&z\end{pmatrix},\overset{\boxed{a}}{\rho}_u]\!]=\begin{pmatrix}z&x\\y&w\end{pmatrix}$$

which implies that

$$[\![\overset{\boxed{a}}{\rho}_u,\begin{pmatrix}w&x\\y&z\end{pmatrix},\overset{\boxed{a}}{\rho}_u]\!]=\begin{pmatrix}w&x\\y&z\end{pmatrix}\Longleftrightarrow(\exists\, k\in F\vartriangle\{0\})\quad\begin{pmatrix}z-x&x\\z+y-x-w&x+w\end{pmatrix}=k\cdot\begin{pmatrix}w&x\\y&z\end{pmatrix}$$

which implies that

$$\{u,v\}^{\bullet}=\{t\in L:(\exists\, r\in F)\quad\overset{\boxed{a}}{\rho}_t=\begin{pmatrix}1&0\\r&1\end{pmatrix}\}. \tag{4}$$

It follows from (3) and (4) that $\overleftrightarrow{u,v}=B\cap a^{\blacklozenge}$. Furthermore, a is in B and, from (4),

$$(B\cap a^{\blacklozenge})^{\bullet}=\{u,v\}^{\bullet}\subset[0\overset{\overset{\boxed{a}}{\rho}}{=\!=}0]=B.$$

Thus (1) and (2) hold for case I.

[Case II: $B=[0\overset{\overset{\boxed{a}}{\rho}}{=\!=}\infty]$ for 0 and M distinct] Choose 1 from M distinct from 0 and ∞ and let $F\equiv M\vartriangle\{\infty\}$.

Then

$$\{\overset{\boxed{a}}{\rho}_x : x\in (B\cap a^{\blacklozenge})\} = \{\begin{pmatrix} 0 & r \\ 1 & 0 \end{pmatrix} : r\in F\}\,. \tag{5}$$

Let $\{u,v\}\subset (B\cap a^{\blacklozenge})$ satisfy $\overset{\boxed{a}}{\rho}_u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\overset{\boxed{a}}{\rho}_v = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$. For $\{w,x,y,z\}\subset F$ we have

$$\lfloor \overset{\boxed{a}}{\rho}_u, \begin{pmatrix} w & x \\ y & z \end{pmatrix}, \overset{\boxed{a}}{\rho}_u \rfloor = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \iff (\exists\, k\in F\triangle\{0\})\ \begin{pmatrix} w & -y \\ -x & z \end{pmatrix} = k\cdot \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

and

$$\lfloor \overset{\boxed{a}}{\rho}_v, \begin{pmatrix} w & x \\ y & z \end{pmatrix}, \overset{\boxed{a}}{\rho}_v \rfloor = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \iff (\exists\, k\in F\triangle\{0\})\ \begin{pmatrix} -w & -y \\ -x & -z \end{pmatrix} = k\cdot \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

which implies that

$$\{u,v\}^{\bullet} = \{t\in L : (\exists\, r\in F)\ \ \overset{\boxed{a}}{\rho}_t = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}\}\,. \tag{6}$$

It follows from (3) and (4) that $\overleftrightarrow{u,v} = B\cap a^{\blacklozenge}$. Furthermore a is not in B but is in$(B\cap a^{\blacklozenge})^{\bullet}$. Thus (1) and (2) hold for case II. Q.E.D.

(8.19) Remark Let $L\equiv\mathbb{L}$ as in Section (7). Then (8.18.1) means that the intersection of the plane trace of a plane tangent to the quadric T with the plane trace of a plane not tangent to T is a line trace.

(8.20) Corollary Let L be a meridian libra with meridian aggregate $\mathcal{T}$. Let K be an element of $L^{\mathbf{lines}}$ and let A be an element of $\mathcal{T}$. If $K\cap A$ has at least two elements, then $K\subset A$.

Proof. Let a be in $K^{\bullet}$ and let $\{b,c\}\subset K\cap A$ be a doubleton. From (8.18.1) follows that $A\cap a^{\blacklozenge}$ is in $L^{\mathbf{lines}}$ and, from (8.17.5) follows that $K = \overleftrightarrow{b,c} = A\cap a^{\blacklozenge}$. Q.E.D.

(8.21) Corollary Let L be a meridian libra with meridian aggregate $\mathcal{T}$. Let A be an element of $\mathcal{T}$. Then $A^{\bullet} = \emptyset$.

Proof. Assume that a were in $a^{\blacklozenge}$. It would follow that $A\cap a^{\blacklozenge} = A$, and so (8.18.1) would imply that A is in $L^{\mathbf{lines}}$. From (8.12.3) would follow that $a\notin A$. Let b be in A. Then $\lfloor a,b,A\rfloor$ would contain a and so would not be A. From (6.3) would follow that A would be abelian. For $\{c,d\}\subset A$ we would have

$$\lfloor d,\lfloor a,b,c\rfloor,d\rfloor = \lfloor\lfloor d,c,b\rfloor,a,d\rfloor = \lfloor\lfloor b,c,d\rfloor,a,d\rfloor = \lfloor b,c,d,d,a\rfloor = \lfloor b,c,a\rfloor = \lfloor a,b,c\rfloor$$

which would mean that $d\in\lfloor a,b,A\rfloor^{\bullet}$. This would imply that $A\subset\lfloor a,b,A\rfloor^{\bullet}$. Since left translates are pairwise disjoint, it would follow that A would have to be $\lfloor a,b,A\rfloor$, which would be absurd. Q.E.D.

(8.22) Theorem Let L be a meridian libra with meridian aggregate $\mathcal{T}$. Let A and B be distinct skew elements of $\mathcal{T}$. Then

$$A\cap B \quad \text{is in} \quad L^{\mathbf{lines}} \tag{1}$$

and

$$(A\cap B)^{\bullet} = (\boxed{A}\wedge\boxed{B})\cap(\boxed{B}\wedge\boxed{A})\,. \tag{2}$$

Proof. Let a be in A. There exist $\infty\in M$ and $n\in N$ such that $A = [\infty \overset{\rho}{=\!=} n]$. We have $\rho_a(\infty) = n$ and so

$$\overset{\boxed{a}}{\rho}_a(\infty) = ({\rho_a}^{-1}\circ\rho_a)(\infty) = ({\rho_a}^{-1})(n) = \infty \implies A = [\infty \overset{\rho}{=\!=} \infty]\,.$$

There exists $\{0,1\}\subset M$ such that $B=[0\overset{\rho}{=\!=}1]$. Since A and b are skew, neither 0 nor 1 can be ∞. We have

$$(\forall x\in A\cap B)(\exists r\in F)\quad \overset{\boxed{a}}{\rho}_x=\begin{pmatrix} r & 1\\ 0 & 1\end{pmatrix}. \tag{3}$$

Let r and s in L satisfy

$$\overset{\boxed{a}}{\rho}_r=\begin{pmatrix} r & 1\\ 0 & 1\end{pmatrix}\quad \text{and}\quad \overset{\boxed{a}}{\rho}_s=\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}.$$

Simple calculations show that a necessary and sufficient condition for $x\in L$ to be in $r^{\bullet}\cap s^{\bullet}$ is for there to exist $u\in F$ such that

$$\overset{\boxed{a}}{\rho}_x=\begin{pmatrix} 1 & u\\ 1 & 0\end{pmatrix}. \tag{4}$$

Thus $\overleftrightarrow{r,s}=\{x\in L:(\exists r\in F)\ \overset{\boxed{a}}{\rho}_x=\begin{pmatrix} r & 1\\ 0 & 1\end{pmatrix}\}$ and

$$(\overleftrightarrow{r,s})^{\bullet}=\{x\in L:(\exists u\in F)\ \overset{\boxed{a}}{\rho}_x=\begin{pmatrix} 1 & u\\ 1 & 0\end{pmatrix}\}. \tag{5}$$

The first of these two equalities implies that $\overleftrightarrow{r,s}=A\cap B$ and is in $L^{\mathbf{lines}}$. The set of all $x\in L$ satisfying (5) is just

$$[\infty\overset{\overset{\boxed{a}}{\rho}}{=\!=}1]\cap[0\overset{\overset{\boxed{a}}{\rho}}{=\!=}\infty]\ =\ (\boxed{A}\wedge\boxed{B})\cap(\boxed{B}\wedge\boxed{A}).$$

From this last and equation (5) follows that

$$(A\cap B)^{\bullet}=(\boxed{A}\wedge\boxed{B})\cap(\boxed{B}\wedge\boxed{A}).$$

Q.E.D.

(8.23) Remarks Let $L\equiv\mathbb{L}$ as in (8.1) and let P and Q be two planes tangent to the quadric T. Then the traces of P and Q are skew if the line $P\cap Q$ of intersection is not tangent to T.

The plane P contains a line P_1 from the regulus $\boxed{T}$ and a line P_2 from the regulus $\boxed{T}$. Similarly, the plane Q contains a line Q_1 from the regulus $\boxed{T}$ and a line Q_2 from the regulus $\boxed{T}$. The content of (8.22.2) is that the line dual to $P\cap Q$ relative to T is just the intersection of the plane containing the lines P_1 and Q_2 with the plane containing the lines P_2 and Q_1.

(8.24) Corollary Let L be a meridian libra with meridian aggregate $\mathcal{T}$. Let A and B be skew elements of $\mathcal{T}$. Then

$$\{x\in L:B=\lfloor x,A,x\rfloor\}=(A\cap B)^{\circ}. \tag{1}$$

Proof. This follows from (5.18.1) and (8.22.2). Q.E.D.

(8.25) Theorem Let L be a meridian libra with meridian aggregate $\mathcal{T}$. Let K be in $L^{\mathbf{lines}}$. Then the four following statements are pairwise equivalent:

$$(\exists A\in\mathcal{T})\quad K\cup K^{\bullet}\subset A, \tag{1}$$

$$(\exists A\in\mathcal{T})\quad K\subset A\quad \text{and}\quad K^{\bullet}\cap A\neq\emptyset, \tag{2}$$

$$(\exists!\, B\in\mathcal{T})\quad K\subset B \tag{3}$$

and $$(\exists\, A \in \mathcal{T}\,) \quad \{E \in \mathcal{T} : K \cap E = \emptyset\} = \{D \in (\boxed{A} \cup \boxed{A}) : D \neq A\}\,. \tag{4}$$

If these four statements hold, then

$$(\forall\, n \in K) \quad K \cap n^{\bullet} = \emptyset\,. \tag{5}$$

Proof. [(1) $\Longrightarrow$ (2)] Trivial.

[(2) $\Longrightarrow$ (3)] Suppose that (2) holds and that a is in $K^{\bullet} \cap A$. Then $K \subset (a^{\bullet} \cap A)$ and so by Theorem (8.18.1), K equals $a^{\bullet} \cap A$. If $\lfloor a,A,a \rfloor$ were not A, they obviously would form a skew pair and so Theorem (8.22.1) would imply that $K = A \cap \lfloor a,A,a \rfloor$, which would imply that $a \in (K \cap K^{\bullet})$, which would violate (8.17.3). It follows that $A = \lfloor a,A,a \rfloor$. Thus, if B were any element of $\mathcal{T}$ distinct from A for which $B \subset K$, then B and A would be a-skew, and so Theorem (8.7.2) would imply that $A \cap B \cap a^{\bullet}$ were a singleton, which would be absurd. It follows that (3) holds.

[(3) $\Longrightarrow$ (4) and (1)] Suppose that (3) holds. Let 0, 1, ∞, q and r be as in Lemma (8.13). Since a is in B and $\overset{\boxed{a}}{\rho}_a$ is the identity mapping, there exists $m \in M$ such that $B = [m \overset{\overset{\boxed{a}}{\rho}}{=} m]$. If m were in F, then for $\{d,e\} \subset M$ such that $e^2 \neq qrd^2$, $\begin{pmatrix} e & rd \\ qd & e \end{pmatrix}(m) = m$. This were absurd, so m must be ∞. It follows that $q = 0 \neq r$. That

$$\{D \in (\boxed{B} \wedge \boxed{B}) : D \neq B\} \subset \{E \in \mathcal{T} : K \cap E = \emptyset\}$$

follows from the fact that $\boxed{B}$ and $\boxed{B}$ are partitions of L. Let $E \in \mathcal{T}$ be disjoint from K and assume that E were neither $\boxed{B}$ nor $\boxed{B}$. Then there would exist $\{j,k\} \subset F$ such that $E = [j \overset{\overset{\boxed{a}}{\rho}}{=} k]$. but

$$\begin{pmatrix} 1 & (\frac{k-j}{r})r \\ (\frac{k-j}{r})q & 1 \end{pmatrix}(j) = \begin{pmatrix} 1 & k-j \\ 0 & 1 \end{pmatrix}(j) = k \Longrightarrow K \cap E \neq \emptyset$$

which would be absurd. It follows that (4) holds. From Lemma (8.10) follows that the image by $\overset{\boxed{a}}{\rho}$ of anything in $K^{\bullet}$ is of the form $\begin{pmatrix} e & -dr \\ 0 & -e \end{pmatrix}$ and so it is evidently in $[\infty \overset{\overset{\boxed{a}}{\rho}}{=} \infty] = B$. Thus (1) holds.

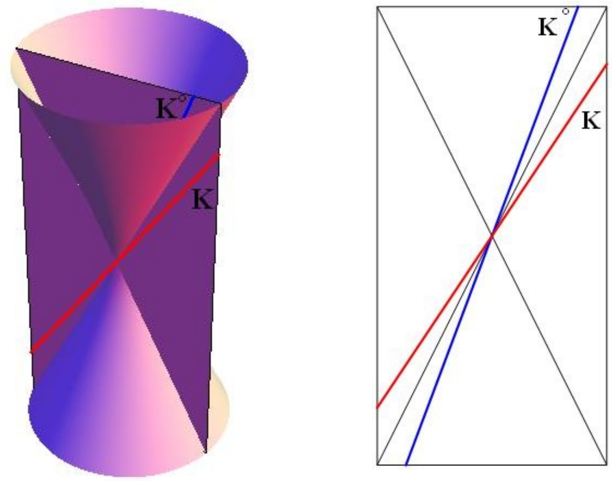
[(4) $\Longrightarrow$ (3)] Trivial.

[(1) $\Longrightarrow$ (5)] Suppose that (1) holds. Since $\boxed{A}$ and $\boxed{A}$ are partitions of L, it is trivial that

$$\{D \in \boxed{A} \cap \boxed{A} : D \neq A\} \subset \{E \in \mathcal{T} : K \cap E = \emptyset\}\,. \tag{6}$$

Suppose that $B \in \mathcal{T}$ is neither a left nor a right translate of A. Let a be an element of $K^{\bullet}$. Since a is in A, it follows that $A = \lfloor a,Ak,a \rfloor$ which implies that A and B are not a-skew. By (8.7.2) we know that $a^{\bullet} \cap A \cap B$ is a singleton. But $a^{\bullet} \cap A = K$, which implies that $K \cap B$ is a singleton. Thus the set containment symbol in (6) can be replaced by an equals symbol. Hence (5) holds. Q.E.D.

(8.26) Definition We shall say that a line trace which satisfies (8.25.3) is **accident**. If a line trace is a subset of more than one element of $\mathcal{T}$, we shall say that it is **incident**. If a line trace is neither accident nor incident, we shall say that it is **excident**.



An Accident Line Trace K with K^{o} Showing Rules of their Plane

(8.27) Theorem Let L be a meridian libra with meridian aggregate $\mathcal{T}$. Let K be a line trace. The following five statements are pairwise equivalent:

$$\text{K is incident}\,, \tag{1}$$

$$\#\{A\in\mathcal{T} : K\subset A\} = 2\,, \tag{2}$$

$$(\exists\,\{A,B\}\subset\mathcal{T}\,)\quad K = A\cap B\,, \tag{3}$$

$$K^{\bullet}\ \text{is incident} \tag{4}$$

and

$$(\exists\,\{A,B\}\subset\mathcal{T}\ \text{skew})\quad K = \{x\in L : \lfloor x,A,x\rfloor = B\,.\} \tag{5}$$

Proof. [(1) $\Longleftrightarrow$ (2)] Suppose that (1) holds. Assume that there existed a tripleton $\{A,B,C\}\subset\mathcal{T}$ of which K were a subset of each element. Then $\boxed{A}$ would be a partition of L of which no element but A were a superset of K. Consequently neither B nor C could be in $\boxed{A}$, whence would follow that $\boxed{A}$ could neither equal $\boxed{B}$ nor $\boxed{C}$. Similarly, $\boxed{B}$ could not be $\boxed{C}$. Since $\Vert\Vert(\mathcal{T}\,)$ is a meridian, it would follow from (1.2.1) that there would exist exactly one element x of L such that $x^{\boxed{\mathcal{T}}}(\boxed{A}) = \boxed{A}$, $x^{\boxed{\mathcal{T}}}(\boxed{B}) = \boxed{B}$ and $x^{\boxed{\mathcal{T}}}(\boxed{C}) = \boxed{C}$. Evidently any element of K could be taken for x in these three equalities and by (8.11.1), K would have at least three distinct elements. This would be absurd, which establishes (2). That (2) implies (1) is trivial.

[(2) $\Longrightarrow$ (3)] Suppose that (2) holds and that A and B are the elements of $\mathcal{T}$ of which K is a subset. From Theorem (8.18.1) follows that $A\cap B$ is a line trace and so, by (8.11.4), it must be K. Thus (3) holds.

[(1) $\Longleftrightarrow$ (4)] Suppose again that (1) holds and let $\{A,B\}\subset\mathcal{T}$ be a doubleton of which K is a subset of each element. From Theorem (8.18) we know that

$$K^{\bullet} = (\boxed{A}\cap\boxed{B})\cap(\boxed{B}\wedge\boxed{A})\,.$$

This implies that K is a subset of both $\boxed{A}\cap\boxed{B}$ and $\boxed{B}\cap\boxed{A}$ and so (4) holds. Since $K = K^{\bullet\bullet}$, we also have that (4) implies (1).

[(1) $\Longrightarrow$ (4)] Suppose that (1) holds and that A and B are the elements of $\mathcal{T}$ each of which K is a subset. Assume that there existed $C\in\mathcal{T}$ such that $K\cap C = \emptyset$ and $K^{\bullet}\not\subset C$. Let a be in $K^{\bullet}$. We saw above that $\lfloor a,A,a\rfloor = B$ and so the pair A and C would be a-skew. It follows from Theorem (8.7.2) that $a^{\blacklozenge}\cap A\cap C$ would be a singleton. But $a^{\blacklozenge}\cap A$ is a superset of K and so by (8.18.1) is precisely K. Thus the singleton

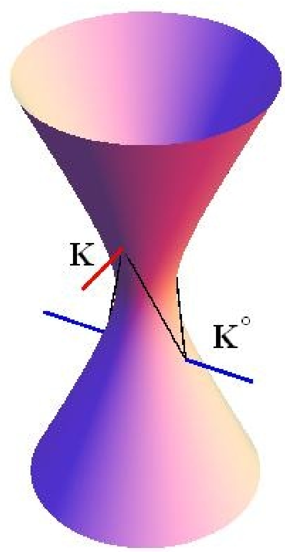
$a^{\bullet}\cap A\cap C$ would be a subset of K, which would be absurd. We have shown that

$$\{C\in\mathcal{T} : K\cap C=\emptyset\}\subset\{D\in\mathcal{T} : K^{\bullet}\subset D\}\,. \tag{6}$$

On the other hand, if $D\in\mathcal{T}$ and $(K^{\bullet}\cup K)\subset D$, then Theorem (8.25.2) would imply that K were accident. It follows that the containment symbol in (6) may be replaced by the equality symbol. This establishes (5)
[(4) $\Longleftrightarrow$ (5)] From Theorem (5.18.2) and Theorem (8.22.2) follows that, for any skew A and B in $\mathcal{T}$,

$$\{x\in L : \lfloor x,A,x\rfloor = B\} = (\boxed{A}\wedge\boxed{B})\cap(\boxed{B}\wedge\boxed{A}) = (A\cap B)^{\bullet}\,.$$

Thus (5) is equivalent to the statement that $K=(A\cap B)^{\bullet}$ for some $\{A,B\}\subset\mathcal{T}$. If $K^{\bullet}$ is incident, then $K^{\bullet}=A\cap B$ for some $\{A,B\}\subset\mathcal{T}$ by (3), which in turn implies that $K=K^{\bullet\bullet}=(A\cap B)^{\bullet}$. On the other hand, if (5) holds, then $K^{\bullet}=(A\cap B)^{\bullet\bullet}=A\cap B$ and so $K^{\bullet}$ is incident. Q.E.D.



An Incident Line Trace K with $K^{\bullet}$

(8.28) Discussion Let K be an incident line trace. Then $K^{\bullet}$ is incident too and so there are precisely two elements A and B of $\mathcal{T}$ which are supersets of $K^{\bullet}$: in fact we have $K^{\bullet}=A\cap B$. Let $C\equiv\boxed{A}\wedge\boxed{B}$ and $D\equiv\boxed{B}\wedge\boxed{A}$. Then C and D are the two elements of $\mathcal{T}$ which are supersets of K: $K=C\cap D$. Thus the incident line traces are the intersections of the skew pairs of $\mathcal{T}$.

(8.29) Theorem Let $\mathcal{T}$ be a meridian aggregate of a properly linear libra L. For K a line trace, the following are pairwise equivalent statements:

$$K \text{ is excident}\,, \tag{1}$$

$$K^{\bullet} \text{ is excident}\,, \tag{2}$$

$$(\forall\, A\in\mathcal{T})\quad K\cap A\neq\emptyset \tag{3}$$

and

$$(\forall\, A\in\mathcal{T})\quad K\cap A \text{ is a singleton.} \tag{4}$$

Proof. [(1) $\Longleftrightarrow$ (2)] Suppose that (1) holds. From (8.25.1) follows that $K^{\bullet}$ cannote be accident. From (8.27.3) we know that $K^{\bullet}$ cannot be incident. Thus (2) holds. That (2) implies (1) follows from interchanging the roles of K and $K^{\bullet}$.

[(3) $\Longrightarrow$ (1)] Suppose that (3) holds. That K cannot be accident follows from (8.25.4). Than K cannot be incident follows from (8.27.4). Thus (1) holds.

[(3) $\Longrightarrow$ (4)] Suppose that (1) holds. Let a and b be distinct points of K. Let 0 be any element of M and let $\infty \equiv \overset{\boxed{a}}{\rho}_b(0)$. Then there exists $\{r,s,l\} \subset F$ such that $\overset{\boxed{a}}{\rho}_b = \begin{pmatrix} r & l \\ s & 0 \end{pmatrix}$. It follows that

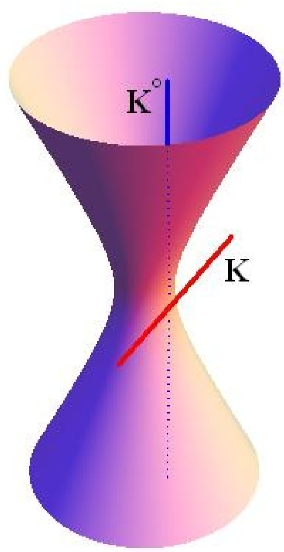
$$K^{\bullet} = \{x \in L : (\exists \{p,q\} \subset F) \quad \overset{\boxed{a}}{\rho}_x = \begin{pmatrix} p & q \\ -qs-pr & -p \end{pmatrix}\}.$$

From this follows that

$$K = K^{\bullet\bullet} = \{x \in L : (\exists \{w,z\} \subset F) \quad \overset{\boxed{a}}{\rho}_x = \begin{pmatrix} w & z \\ sz & w-rz \end{pmatrix}\}.$$

Let A be any element of $\mathcal{T}$. Then there exists $\{m,n\} \subset M$ such that $A = [m \overset{\boxed{a}}{\underset{\rho}{=}} n]$. Evidently a is in $[m \overset{\boxed{a}}{\underset{\rho}{=}} n]$ whenever $m = n$. If any other $k \in K$ were in $[m \overset{\boxed{a}}{\underset{\rho}{=}} n]$ for $m = n$, then all of K would be as well, and so K would not be excident. Thus we may and shall presume that $m \neq n$. If $m = \infty$, we set $z \equiv l$ and $w \equiv l + rn$ to obtain $x \in [m \overset{\boxed{a}}{\underset{\rho}{=}} n]$. If $n = \infty$, we set $w \equiv l$ and $z \equiv -m$ to obtain $x \in [m \overset{\boxed{a}}{\underset{\rho}{=}} n]$. If neither m nor n is ∞, we set $z \equiv m - n$ and $w \equiv sm - rn - l$ to obtain $x \in [m \overset{\boxed{a}}{\underset{\rho}{=}} n]$. It follows that (4) holds.

[(4) $\Longrightarrow$ (3)] Trivial. Q.E.D.



An Excident Line Trace K with $K^{\bullet}$

(8.30) Theorem Let L be a meridian libra with meridian aggregate $\mathcal{T}$. Let a be in L and b be in $a^{\blacklozenge}$. Then

$$b^{\blacklozenge} \cap a^{\blacklozenge} \text{ is a line trace}, \tag{1}$$

$$(\forall c \in (b^{\blacklozenge} \cap a^{\blacklozenge})) \quad c^{\blacklozenge} \cap b^{\blacklozenge} \cap a^{\blacklozenge} \text{ is a singleton } \{d\} \text{ and } b = \lfloor c,a,d \rfloor, \tag{2}$$

$$\text{if } \overleftrightarrow{a,b} \text{ is incident, } (\exists\, c\in(b^{\bullet}\cap a^{\bullet}))(\exists\, d\in(c^{\bullet}\cap b^{\bullet}\cap a^{\bullet}))\quad \overleftrightarrow{a,c} \text{ and } \overleftrightarrow{c,d} \text{ are incident and } b=\lfloor c,a,d\rfloor \tag{3}$$

and

$$\text{if } \overleftrightarrow{a,b} \text{is excident, } (\exists\, c\in(b^{\bullet}\cap a^{\bullet}))(\exists\, d\in(c^{\bullet}\cap b^{\bullet}\cap a^{\bullet}))\quad \overleftrightarrow{a,c} \text{ is incident, } \overleftrightarrow{c,d} \text{ is excident and } b=\lfloor c,a,d\rfloor\,. \tag{4}$$

Proof. Let 0 be a point in M not fixed by $\overset{a}{\rho}_b$. Let $\infty\equiv\overset{a}{\rho}_b(0)$ and choose $1\in M$ distinct from 0 and ∞. Then here exists $e\in M$ such that

$$\overset{a}{\rho}_b=\begin{pmatrix}0&1\\e&0\end{pmatrix}. \tag{5}$$

It follows that

$$\{\overset{a}{\rho}_x : x\in(a^{\bullet}\cap b^{\bullet})\}=\{\begin{pmatrix}x&y\\-ey&-x\end{pmatrix}:\{x,y\}\subset F\}\,. \tag{6}$$

This implies that

$$b^{\bullet}\cap a^{\bullet}=\overleftrightarrow{c,d} \text{ where } \overset{a}{\rho}_c=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \text{ and } \begin{pmatrix}0&1\\-e&0\end{pmatrix}$$

which establishes (1). That (2) holds now follows from direct calculation.

If $\overleftrightarrow{a,b}$ is incident, then it is a subset of two elements A and B of $\mathcal{T}$ and so in particular $\{a,b\}\subset A\cap B$: there exists m and 1 in M distinct such that $\overset{a}{\rho}_b(1)=1$ and $\overset{a}{\rho}_b(m)=m$. Thus

$$\overset{a}{\rho}_b=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \overset{a}{\rho}_c=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \text{ and } \overset{a}{\rho}_d=\begin{pmatrix}0&1\\-e&0\end{pmatrix}.$$

Evidently $\overleftrightarrow{a,c}\subset([0\overset{\overset{a}{\rho}}{=\!=}0]\cap[\infty\overset{\overset{a}{\rho}}{=\!=}\infty])$ and so $\overleftrightarrow{a,c}$ is incident. In addition $\overleftrightarrow{c,d}\subset[1\overset{\overset{a}{\rho}}{=\!=}-1]\cap[-1\overset{\overset{a}{\rho}}{=\!=}1]$, which implies that $\overleftrightarrow{c,d}$ is incident. Thus (3) holds.

If $\overleftrightarrow{a,b}$ is excident, then it has no superset in $\mathcal{T}$. In particular, the equation $t=\overset{a}{\rho}_b(t)=\frac{1}{et}$ has no solution for t. From equation (2) we know that $\overset{a}{\rho}_c$ is of the form $\begin{pmatrix}x&y\\-ey&-x\end{pmatrix}$ and $\overset{a}{\rho}_d$ is of the form $\begin{pmatrix}r&s\\-es&-r\end{pmatrix}$. The equation $\overset{a}{\rho}_d(t)=\overset{a}{\rho}_c(t)$ resolves into $et^2=-1$. Hence the line trace $\overleftrightarrow{c,d}$ is excident as well. If we choose c to be such that $\overset{a}{\rho}_c=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, then evidently $\overleftrightarrow{a,c}\subset([0\overset{\overset{a}{\rho}}{=\!=}0]\cap[\infty\overset{\overset{a}{\rho}}{=\!=}\infty])$ and so $\overleftrightarrow{a,c}$ is incident. This establishes (4). QED

(8.31) Theorem Let L be a meridian libra with meridian aggregate $\mathcal{T}$. Let K be a line trace and let a be an element of K. Then

$$K\cap a^{\bullet}\neq\emptyset \iff K \text{ is not accident.} \tag{1}$$

Proof. [$\Longrightarrow$] This follows from (8.25.5).

[$\Longleftarrow$] Let $b\in K$ be distinct from a and assume that K were not accident. If b were in $a^{\bullet}$, we would be done, so we shall presume that b is not in $a^{\bullet}$. By Lemma (8.10) there would exist an a-representation $\overset{a}{\rho}$, a choice of basis, and $\{q,r\}\subset F$ such that

$$K=\{\begin{pmatrix}e&rd\\qd&e\end{pmatrix}:\{e,d\}\subset F\}\quad\text{and}\quad\overset{a}{\rho}_b=\begin{pmatrix}1&r\\q&1\end{pmatrix}.$$

Evidently an element of K is in $a^{\bullet}$ if and only if it is of the form $\begin{pmatrix}0&r\\q&0\end{pmatrix}$. For this it is necessary and sufficient that $q\neq0\neq r$. But one easily checks that

$$\mathrm{q}=0 \Longrightarrow \mathrm{K}\subset[\infty \overset{\overset{\boxed{a}}{\rho}}{=\!=} \infty] \quad \text{and} \quad \mathrm{r}=0 \Longrightarrow \mathrm{K}\subset[0 \overset{\overset{\boxed{a}}{\rho}}{=\!=} 0]$$

both of which would be absurd, since we have assumed K not to be accident. Q.E.D.

(8.32) Remarks Let $\mathbb{L}$ be as in (8.1), let K be a line in $\mathbb{E}$ and let M be its dual line $\bigcap_{\mathrm{x}\in\mathrm{K}} \tau(\mathrm{x})$.

Then $\widehat{\mathrm{K}}$ is incident if K intersects T in two separate points. Theorem (8.27.4) states that, if this is true, then $\widehat{\mathrm{M}}$ is also incident. Theorem (8.27.4) states that $\widehat{\mathrm{K}}$ is incident if and only if K is the intersection of two planes tangent to T of which the intersection is not a line contained in T .

The line trace $\widehat{\mathrm{K}}$ is excident if K is disjoint from T . Theorem (8.29.2) states that, if this is true, then $\widehat{\mathrm{M}}$ is also excident. It follows from (8.29.3) that $\widehat{\mathrm{K}}$ is excident if, and only if, K intersects each tangent plane to T in a point not in T .

The line trace $\widehat{\mathrm{K}}$ is accident if K is tangent to T at a single point. Since for incident $\widehat{\mathrm{K}}$, $\widehat{\mathrm{M}}$ is also incident, and for excident $\widehat{\mathrm{K}}$, $\widehat{\mathrm{M}}$ is also excident, it follows logically that if $\widehat{\mathrm{K}}$ is accident, then $\widehat{\mathrm{M}}$ must also be accident. Theorem (8.31) states that $\widehat{\mathrm{K}}$ is accident if, and only if, there exists a plane in $\mathbb{E}$ not tangent to T of which the intersection with K is void.

(8.33) Theorem Let L be a meridian libra with meridian aggregate $\mathcal{T}$. Let a and b be distinct elements of L. Then

$$\overleftrightarrow{\mathrm{a,b}} \text{ is excident} \Longleftrightarrow (\exists\, \{\mathrm{c,d}\}\subset \mathrm{a}^{\blacklozenge}) \quad \overleftrightarrow{\mathrm{a,c}} \text{ is incident, } \overleftrightarrow{\mathrm{c,d}} \text{ is excident, and } \mathrm{b}=\lfloor \mathrm{c,a,d}\rfloor , \tag{1}$$

$$\overleftrightarrow{\mathrm{a,b}} \text{ is incident} \Longleftrightarrow (\exists\, \{\mathrm{c,d}\}\subset \mathrm{a}^{\blacklozenge}) \quad \overleftrightarrow{\mathrm{a,c}} \text{ and } \overleftrightarrow{\mathrm{c,d}} \text{ are incident and } \mathrm{b}=\lfloor \mathrm{c,a,d}\rfloor \tag{2}$$

and
$$\overleftrightarrow{\mathrm{a,b}} \text{ is accident} \Longleftrightarrow (\exists\, \{\mathrm{c,d}\}\subset \mathrm{a}^{\blacklozenge}) \quad \overleftrightarrow{\mathrm{a,c}} \text{ is incident, } \overleftrightarrow{\mathrm{c,d}} \text{ is excident and } \mathrm{b}=\lfloor \mathrm{c,a,d}\rfloor . \tag{3}$$

Proof. Suppose first that b is in $\mathrm{a}^{\blacklozenge}$. If $\overleftrightarrow{\mathrm{a,b}}$ were accident, then Thereom (8.25.5) would imply that $\mathrm{b}\in(\mathrm{a}^{\blacklozenge}\cap \overleftrightarrow{\mathrm{a,b}})=\varnothing$: an absurdity. Thus $\overleftrightarrow{\mathrm{a,b}}$ is either excident or incident, and so (2) and (3) follow from Theorem (8.30.4) and (8.30.3).

For the remainder of this proof we shall presume that b is not in $\mathrm{a}^{\blacklozenge}$. Suppose that $\overleftrightarrow{\mathrm{a,b}}$ is incident. Then $\overleftrightarrow{\mathrm{a,b}}$ is contained in two distince elements of $\mathcal{T}$, which means that $\overset{\boxed{a}}{\rho}_{\mathrm{a}}$ and $\overset{\boxed{a}}{\rho}_{\mathrm{b}}$ agre on two distinct elements of M . This means that $\overset{\boxed{a}}{\rho}_{\mathrm{b}}$ has two fixed points, and so is a dilation. By (1.2.3) there exist $\{\pi,\sigma\}$ $\mathcal{M}$ agreeing on two distinct points of M and such that π is a dilation and such that $\overset{\boxed{a}}{\rho}_{\mathrm{b}}=\pi\circ\sigma$. Choose $\{\mathrm{c,d}\}\subset \mathrm{a}^{\blacklozenge}$ such that $\overset{\boxed{a}}{\rho}_{\mathrm{c}}=\pi$ and $\overset{\boxed{a}}{\rho}_{\mathrm{d}}=\sigma$. We have

$$\llbracket \overset{\boxed{a}}{\rho}_{\mathrm{c}}, \overset{\boxed{a}}{\rho}_{\mathrm{a}}, \overset{\boxed{a}}{\rho}_{\mathrm{d}} \rrbracket = \overset{\boxed{a}}{\rho}_{\mathrm{c}} \circ \overset{\boxed{a}}{\rho}_{\mathrm{d}} = \pi\circ\sigma = \overset{\boxed{a}}{\rho}_{\mathrm{b}} \Longrightarrow \mathrm{b}=\lfloor \mathrm{c,a,d}\rfloor . \tag{4}$$

Since $\overset{\boxed{a}}{\rho}_{\mathrm{c}}=\pi$ has two fixed points, it agrees with $\overset{\boxed{a}}{\rho}_{\mathrm{a}}$ at two points — hence the line trace $\overleftrightarrow{\mathrm{a,c}}$ is incident. Since $\overset{\boxed{a}}{\rho}_{\mathrm{c}}$ and $\overset{\boxed{a}}{\rho}_{\mathrm{d}}$ agree on two distinct points, it follows that $\overleftrightarrow{\mathrm{c,d}}$ is incident. This establishes (2).

Now suppose that $\overleftrightarrow{\mathrm{a,b}}$ is excident. This means that a and b are in no common element of $\mathcal{T}$: that $\overset{\boxed{a}}{\rho}_{\mathrm{b}}$ leaves no point fixed, and thus is a pure rotation. By (1.2.4) there exists $\{\pi,\sigma\}\{\mathcal{M}\}$ agreeing on no point of M , such that π is dilation and such that $\overset{\boxed{a}}{\rho}_{\mathrm{b}}=\pi\circ\sigma$. Choose $\{\mathrm{c,d}\}\subset \mathrm{a}^{\blacklozenge}$ such that $\overset{\boxed{a}}{\rho}_{\mathrm{c}}=\pi$ and $\overset{\boxed{a}}{\rho}_{\mathrm{d}}=\sigma$. As before, it follows that (1) holds and that $\overleftrightarrow{\mathrm{a,c}}$ is incident. This time however, $\overleftrightarrow{\mathrm{c,d}}$ is excident since $\overset{\boxed{a}}{\rho}_{\mathrm{c}}$ and $\overset{\boxed{a}}{\rho}_{\mathrm{d}}$ agree on no point of M . This establishes (1).

Finally, we suppose that $\overleftrightarrow{\mathrm{a,b}}$ is accident. This means that a and b are in a single element of $\mathcal{T}$: that $\overset{\boxed{a}}{\rho}_{\mathrm{a}}$ and $\overset{\boxed{a}}{\rho}_{\mathrm{b}}$ agree on a single point of M so that $\overset{\boxed{a}}{\rho}_{\mathrm{b}}$ has exactly one fixed point and is thus a translation. By (1.2.2) there exist $\{\pi,\sigma\}\subset\mathcal{M}$ with a single common fixed point of M such that π and σ are dilations and such that $\overset{\boxed{a}}{\rho}_b = \pi\circ\sigma$. Choose $\{c,d\}\subset a^{\blacklozenge}$ such that $\overset{\boxed{a}}{\rho}_c = \pi$ and $\overset{\boxed{a}}{\rho}_d = \sigma$. As before, it follows that (1) holds and that $\overleftrightarrow{a,c}$ is incident. This time however, $\overleftrightarrow{c,d}$ is accident since $\overset{\boxed{a}}{\rho}_c$ and $\overset{\boxed{a}}{\rho}_d$ agree on a single point of M. This establishes (3). Q.E.D.

(8.34) Corollary Let L be a meridian libra with meridian aggregate $\mathcal{T}$. Let a be an element of L. Then

$$Ł = \{\lfloor x,y,z \rfloor : \{x,y,z\}\subset a^{\blacklozenge}\}. \tag{1}$$

Proof. By Theorem (8.30) there exist $\{a,b,c\}\subset a^{\blacklozenge}$ such that $a = \lfloor d,b,c \rfloor$. That (1) holds now follows from Theorem (8.33). Q.E.D.

(8.35) Purpose The followin two theorems are for use in a later paper.

(8.36) Theorem Let L be a meridian libra with meridian aggregate $\mathcal{T}$. Let A be an element of $\mathcal{T}$ and K and V be lines in $\mathbb{E}$ such that $(K\cup V)\subset A$. Then either $K\cap V\neq\emptyset$ or

$$(\exists X\in(\boxed{A}\cap\boxed{A}))\quad (K^{\bullet}\cup V^{\bullet})\subset X. \tag{1}$$

Proof. Let a be any element of K. Where μ is defined as in (8.10.1), we let $\infty\equiv\mu(\boxed{A})$ and $F\equiv M\triangle\{\infty\}$. We have

$$\rho_a(\infty) = \nu(a^{\boxed{\mathcal{T}}}(\mu^{-1}(\infty))) = \nu(a^{\boxed{\mathcal{T}}}(\boxed{A})) = \nu(\boxed{A})$$

and so

$$[\infty \overset{\overset{\boxed{a}}{\rho}}{=\!=} \infty] = (\mu^{-1}(\infty))\wedge(\mu^{-1}(\rho_a\infty)) = \boxed{A}\wedge\boxed{A} = A.$$

Let b be an element of K distinct from a and let c be an element of V not in K. Then $c\neq a$ and so there exists some point $0\in F$ such that $1\equiv\overset{\boxed{a}}{\rho}_c\neq 0$. Let $s\equiv\overset{\boxed{a}}{\rho}_b(0)$ and $r\equiv\overset{\boxed{a}}{\rho}_b(1)-s$. It follows that

$$\overset{\boxed{a}}{\rho}_c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad\text{and}\quad \overset{\boxed{a}}{\rho}_b = \begin{pmatrix} r & s \\ 0 & 1 \end{pmatrix}.$$

If $r = 1$, then $\overset{\boxed{a}}{\rho}_a$ and $\overset{\boxed{a}}{\rho}_b$ agree only at the one point ∞ of M, which implies that K is accident. Direct calculation shows that

$$\{a,b\}^{\bullet} = \{x\in L : (\exists\, t\in F)\quad \overset{\boxed{a}}{\rho}_x = \begin{pmatrix} -1 & t \\ 0 & 1 \end{pmatrix}\} \quad\text{and}\quad K = \overleftrightarrow{a,b} = \{x\in L : (\exists\, t\in F)\quad \overset{\boxed{a}}{\rho}_x = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\}. \tag{2}$$

If $r\neq 1$, then $\overset{\boxed{a}}{\rho}_a$ and $\overset{\boxed{a}}{\rho}_b$ agree also at the point $\frac{s}{1-r}$ and so K is incident with

$$K = \{x\in L : (\exists\, t\in F)\quad \overset{\boxed{a}}{\rho}_x = \begin{pmatrix} t+(1-t)r & (1-t)s \\ 0 & 1 \end{pmatrix}\}. \tag{3}$$

Recalling that $1 = \overset{\boxed{a}}{\rho}_c(0)$, we choose $q\in F$ such that $\overset{\boxed{a}}{\rho}_c = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}$. Let d be an element of V distinct from c and choose $\{u,v\}\subset F$ such that $\overset{\boxed{a}}{\rho}_d = \begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix}$. If $q = u$, arguing as above we find that V is parabolic and that

$$V = \overleftrightarrow{c,d} = \{x\in L : (\exists\, t\in F)\quad \overset{\boxed{a}}{\rho}_x = \begin{pmatrix} q & t \\ 0 & 1 \end{pmatrix}\}. \tag{4}$$

If $q \neq u$, then $\overset{\boxed{a}}{\rho}_c$ and $\overset{\boxed{a}}{\rho}_d$ agree also at $\frac{v-1}{q-u}$ and so V is incident and

$$\overleftrightarrow{c,d} = \{x \in L : (\exists\, t \in F) \quad \overset{\boxed{a}}{\rho}_x = \begin{pmatrix} tq+(1-t)u & t+(1-t)v \\ 0 & 1 \end{pmatrix}\}. \tag{5}$$

We have four cases, which we shall examine *in seriatim*:

[Case I: ((2) and (4) hold]. In this case (1) holds where $X \equiv A$.

[Case II: ((2) and (5) hold]. The element $y \in L$ for which $\overset{\boxed{a}}{\rho}_y = \begin{pmatrix} 1 & \frac{1-u}{q-u} \\ 0 & 1 \end{pmatrix}$ is in both K and V.

[Case III: ((3) and (4) hold]. Setting $t \equiv \frac{q-r}{1-r}$, we have $\begin{pmatrix} q & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t+(1-t)r & (1-t)s \\ 0 & 1 \end{pmatrix}$ and so this common value is in both K and V.

[Case IV: ((3) and (5) hold]. We break this case into two subcases:

[Subcase I: $\frac{s}{1-r} = \frac{v-1}{q-u}$] If $\overset{\boxed{a}}{\rho}_c(\frac{v-u}{q-u})$ were equal to $\frac{v-u}{q-u}$, then

$$K = [\infty \overset{\overset{\boxed{a}}{\rho}}{=\!=} \infty] \cap [\tfrac{s}{1-r} \overset{\overset{\boxed{a}}{\rho}}{=\!=} \tfrac{s}{1-r}] = [\infty \overset{\overset{\boxed{a}}{\rho}}{=\!=} \infty] \cap [\tfrac{v-1}{q-u} \overset{\overset{\boxed{a}}{\rho}}{=\!=} \tfrac{v-1}{q-u}] = V$$

which were absurd. Thus $e \neq f$ where $e \equiv \frac{v-1}{q-u}$ and $f \equiv \overset{\boxed{a}}{\rho}_c(\frac{v-1}{q-u})$. We have

$$K = [\infty \overset{\overset{\boxed{a}}{\rho}}{=\!=} \infty] \cap [e \overset{\overset{\boxed{a}}{\rho}}{=\!=} e] \quad \text{and} \quad V = [\infty \overset{\overset{\boxed{a}}{\rho}}{=\!=} \infty] \cap [e \overset{\overset{\boxed{a}}{\rho}}{=\!=} f]$$

which implies

$$K^{\bullet} = [\infty \overset{\overset{\boxed{a}}{\rho}}{=\!=} e] \cap [e \overset{\overset{\boxed{a}}{\rho}}{=\!=} \infty] \quad \text{and} \quad V = [\infty \overset{\overset{\boxed{a}}{\rho}}{=\!=} f] \cap [e \overset{\overset{\boxed{a}}{\rho}}{=\!=} \infty]$$

which in turn implies (1) for $X \equiv [e \overset{\overset{\boxed{a}}{\rho}}{=\!=} \infty]$.

[Subcase II: $\frac{s}{1-r} \neq \frac{v-1}{q-u}$] Letting $e \equiv i - v + \frac{q-u}{1-r}$, we see that $e \neq 0$. We now define

$$t_2 \equiv \frac{\frac{u-r}{1-r}s + s - v}{e} \quad \text{and} \quad t_1 \equiv \frac{t_2(q-u)+u-r}{q-u}$$

and obtain

$$\begin{pmatrix} t_1+(1-t_1)r & (1-t_1)s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t_2q+(1-t_2)u & t_2+(1-t_2)v \\ 0 & 1 \end{pmatrix}.$$

Thus, in this subcase, K and V have a common element. Q.E.D.

(8.37) Theorem Let L be a meridian libra with meridian aggregate $\mathcal{T}$. Let a be an element of L and let K and V be distinct line traces contained in $a^{\blacklozenge}$. Then either $K \cap V \neq \emptyset$ or

$$(\exists\, X \in \mathcal{T}) \quad (K^{\bullet} \cup V^{\bullet}) \subset X. \tag{1}$$

Proof. The line trace $K^{\bullet}$ contaios a and so, by (8.10), there exists a choice $\{0,1,\infty\}$ of basis for M and a subset $\{q,r\}$ of $F \equiv M \triangle \{\infty\}$ such that

$$\{\overset{\boxed{a}}{\rho}_x : x \in L\} = \{\begin{pmatrix} x & -ry \\ qy & -x \end{pmatrix} : x,y \in F \text{ and } x^2 \neq qry^2\}. \tag{2}$$

Let c be an lement of $K^{\bullet}$ distinct from a and choose $\{t,u,w,v\} \subset F$ such that $\overset{\boxed{a}}{\rho}_c = \begin{pmatrix} t & u \\ v & w \end{pmatrix}$. Since $V = \{a,c\}^{\bullet}$, direct computation shows that, if $t \neq w$, then

$$\{\overset{\boxed{a}}{\rho}_x : x\in V\} = \{\begin{pmatrix} uy+vx & (t-w)x \\ (t-w)y & -uy-vx \end{pmatrix} : \{x,y\}\subset F \quad \text{and} \quad (uy+vw)^2+(t-w)^2xy \neq 0\}\,. \tag{3}$$

If $(uq+vr)^2 \neq rq(w-t)^2$, then there exists $d\in L$ such that

$$\overset{\boxed{a}}{\rho}_d = \begin{pmatrix} uq+vr & -r(w-t) \\ q(w-t) & -(qr+vr) \end{pmatrix}.$$

Direct computation with (2) and (3) show that d is in $K\cap V$. If $(uq+vr)^2 = rq(w-t)^2$, we have the following cases:$t\neq w$ and $t=w$ and, for the first of these two cases, three subcases: $q=0$, $r=0$ and $qr\neq 0$.

[Case $t\neq w$ and $q=0$] Since $(uq+vr)^2 = rq(w-t)^2$, we have $v=0$ as well. This implies that

$$\overset{\boxed{a}}{\rho}_c(\infty) = \infty\,,$$

whence follows that $V^{\bullet}[\infty \overset{\overset{\boxed{a}}{\rho}}{=} \infty]$. If $c'\in L$ is such that $\overset{\boxed{a}}{\rho}_{c'} = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$, then the fact that $q=0$ implies that c' is in $K^{\bullet}$. From

$$\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}(\infty) = \infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \overset{\boxed{a}}{\rho}_a$$

follows that $K\subset[\infty \overset{\overset{\boxed{a}}{\rho}}{=} \infty]$.

[Case $t\neq w$ and $r=0$] Since $(uq+vr)^2 = rq(w-t)^2$, it follows that $u=0$. This implies that $\overset{\boxed{a}}{\rho}_c(0) = 0$, whence follows that $V^{\bullet}\subset[\infty \overset{\overset{\boxed{a}}{\rho}}{=} \infty]$. If $k\in L$ is such that $\overset{\boxed{a}}{\rho}_k = \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}$, then the fact that $r=0$ implies that k is in $K^{\bullet}$. Then, since

$$\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}(0) = 0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \overset{\boxed{a}}{\rho}_a\,,$$

it follows that $K\subset[0 \overset{\overset{\boxed{a}}{\rho}}{=} 0]$.

[Case $t\neq w$ and $rq\neq 0$] From $(uq+vr)^2 = rq(w-t)^2$ follows that

$$\frac{r}{q} = (q\frac{uq+vr}{w-t})^2\,.$$

Let $e\in L$ satisfy $\overset{\boxed{a}}{\rho}_e = \begin{pmatrix} 0 & r \\ q & 0 \end{pmatrix}$. Then equation $\overset{\boxed{a}}{\rho}_e(x) = x$ for $x\in F$ is the same as the equation $x^2 = \frac{r}{q}$. Thus it has the solutions

$$x = \pm\frac{1}{q}(\frac{uq+vr}{w-t})\,.$$

In particular we have

$$e\in[\frac{1}{q}(\frac{uq+vr}{w-t}) \overset{\overset{\boxed{a}}{\rho}}{=} \frac{1}{q}(\frac{uq+vr}{w-t})]\,,$$

which, since a is there as well, implies that

$$V^{\bullet}\subset[\frac{1}{q}(\frac{uq+vr}{w-t}) \overset{\overset{\boxed{a}}{\rho}}{=} \frac{1}{q}(\frac{uq+vr}{w-t})]\,.$$

On the other hand we have

$$\begin{pmatrix} t & u \\ v & w \end{pmatrix}(\frac{1}{q}(\frac{uq+vr}{w-t})) = \frac{1}{q}(\frac{uq+vr}{w-t}) \iff \frac{1}{q}(\frac{uq+vr}{w-t})+u = (\frac{v}{q}\frac{uq+vr}{w-t})^2 + \frac{w}{q}\frac{uq+vr}{w-t} \iff$$

$$v\frac{r}{q} = \frac{w}{q}\frac{uq+vr}{w-t} - \frac{t}{q}\frac{uq+vr}{w-t} - u \iff v\frac{r}{q} = \frac{uq+vr}{q} - \frac{uq}{q}$$

which last is obviously true. It follows that $\overset{\boxed{a}}{\rho}_c$ is in$[\frac{1}{q}(\frac{uq+vr}{w-t}) \overset{\overset{\boxed{a}}{\rho}}{=} \frac{1}{q}(\frac{uq+vr}{w-t})]$. Since $\overset{\boxed{a}}{\rho}_a$ is as well, we have $V^{\bullet} \subset [\frac{1}{q}(\frac{uq+vr}{w-t}) \overset{\overset{\boxed{a}}{\rho}}{=} \frac{1}{q}(\frac{uq+vr}{w-t})]$.

[Case $t = w$] Choose d in L such that $\overset{\boxed{a}}{\rho}_d = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. From (2) we know that d is in K. Direct calculation shows that d is in $c^{\blacklozenge}$, and so, since it evidently is in $a^{\blacklozenge}$ as well, d is in V. Q.E.D.

9. Product Libras

(9.1) Discussion, Definitions and Notation We here return to that question raised in (8.9.5) for the circular meridian libra, but which will be investigated here for general meridian libras.

Let L be a properly linear libra. The injection

$$L \ni x \hookrightarrow [x,x] \in L \times L$$

of L into its symmetrization[17] libra $L \times L$ is only an isomorphism if L is abelian. When L is non-abelian, the subset $\{[x,x] : x \in L\}$ is not balanced, but one may consider the intersection of all balanced supersets which are subsets of $L \times L$. We shall denote that set by

$$L \boxtimes L \equiv \{ \lfloor\!\lceil [x_1,x_1],\ldots,[x_{2k\text{-}1},x_{2k\text{-}1}] \rceil\!\rfloor : x \in L^{\underline{2k\text{-}1}} \text{ and } k \in \mathbb{N} \}. \tag{1}$$

Using the definition of $\lfloor\!\lceil ,, \rceil\!\rfloor$, (1) may be simplified to

$$L \boxtimes L \equiv \{ [\lfloor x_1,\ldots,x_{2k\text{-}1} \rfloor, \lfloor x_{2k\text{-}1},\ldots,x_1 \rfloor] : x \in L^{\underline{2k\text{-}1}} \text{ and } k \in \mathbb{N} \}. \tag{2}$$

Reversing the order of the elements of the sequence in (2), one sees that

$$L \boxtimes L \equiv \{ [x,y] : [y,x] \in L \boxtimes L \}. \tag{3}$$

Related to $L \boxtimes L$ is the subgroup of $L^{L}!$ generated by $\{ {}_x\pi_x : x \in L \}$:[18]

$$L \circledast L \equiv \{ {}_{x_1}\pi_{x_1} \circ \ldots \circ {}_{x_{2k\text{-}1}}\pi_{x_{2k\text{-}1}} : x \in L^{\underline{2k\text{-}1}} \text{ and } k \in \mathbb{N} \}. \tag{4}$$

Expanding (4), one obtains

$$L \circledast L \equiv \{ {}_{\lfloor x_1,\ldots,x_{2k\text{-}1} \rfloor}\pi_{\lfloor x_{2k\text{-}1},\ldots,x_1 \rfloor} : x \in L^{\underline{2k\text{-}1}} \text{ and } k \in \mathbb{N} \}. \tag{5}$$

(9.2) Theorem For any libra L, $L \boxtimes L$ is a normal balanced subset of $L \times L$.

Proof. For $\{m,n,l\} \subset L$

$$\lfloor\!\lceil [m,n],[l,l],L \boxtimes L \rceil\!\rfloor =$$

$$\{ \lfloor\!\lceil [m,n],[l,l],[x_1,x_1],\ldots,[x_{2k\text{-}1},x_{2k\text{-}1}] \rceil\!\rfloor : x \in L^{\underline{2k\text{-}1}} \text{ and } k \in \mathbb{N} \} =$$

$$\{ [\lfloor m,l,x_1,\ldots,x_{2k\text{-}1} \rfloor, \lfloor x_{2k\text{-}1},\ldots,x_1,l,m \rfloor] : x \in L^{\underline{2k\text{-}1}} \text{ and } k \in \mathbb{N} \} =$$

$$\{ [\lfloor m,l,x_1,l,m,m,l,\ldots,m,x_{2k\text{-}1},l,m,m,l \rfloor, \lfloor l,n,n,l,x_{2k\text{-}1},l,n,\ldots,l,n,n,l,x_1,l,l \rfloor] : x \in L^{\underline{2k\text{-}1}} \text{ and } k \in \mathbb{N} \} =$$

$$\{ [\lfloor \lfloor m,y_1,m \rfloor,\ldots,\lfloor m,y_{2k\text{-}1},m \rfloor,m,l \rfloor, \lfloor l,n,\lfloor n,y_{2k\text{-}1},n \rfloor,\ldots,\lfloor n,y_1,n \rfloor \rfloor] : y \in L^{\underline{2k\text{-}1}} \text{ and } k \in \mathbb{N} \} =$$

$$\{ [\lfloor z_1,\ldots,z_{2k\text{-}1},m,l \rfloor, \lfloor l,n,z_{2k\text{-}1},\ldots,z_1 \rfloor] : z \in L^{\underline{2k\text{-}1}} \text{ and } k \in \mathbb{N} \} =$$

$$\{ \lfloor\!\lceil [x_1,x_1],\ldots,[x_{2k\text{-}1},x_{2k\text{-}1}],[m,n],[l,l] \rceil\!\rfloor : x \in L^{\underline{2k\text{-}1}} \text{ and } k \in \mathbb{N} \} =$$

$$\lfloor\!\lceil L \boxtimes L,[m,n],[l,l] \rceil\!\rfloor.$$

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[17] The symmetrization is defined in (4.11.3).

[18] As in (6.1.4), for $\{a,b\} \subset L$, ${}_a\pi_b \big| L \ni x \hookrightarrow \lfloor a,x,b \rfloor \in L$.

We have shown that every left coset of $L\boxtimes L$ is a right coset of $L\boxtimes L$. Q.E.D.

(9.3) Theorem If $L\boxtimes L = L\times L$, then

$$(\forall\, \{x,y\}\subset L)(\exists\, \phi\in L\circledast L)\quad \phi(x) = y\,. \tag{1}$$

Proof. Let m and n be in L. By hypothesis and (9.1.2) there exist $k\in\mathbb{N}$ and $x\in L^{\underline{2k-1}}$ such that

$$[m,n] = [\,\lfloor x_1,\ldots,x_{2k-1}\rfloor,\lfloor x_{2k-1},\ldots,x_1\rfloor\,]\,.$$

Consequently

$$_{\lfloor x_1,\ldots,x_{2k-1}\rfloor}\pi_{\lfloor x_{2k-1},\ldots,x_1\rfloor}(n) = \lfloor x_1,\ldots,x_{2k-1},\lfloor x_{2k-1},\ldots,x_1\rfloor,x_k,\ldots,x_1\rfloor =$$
$$\lfloor x_1,\ldots,x_{2k-1}x_1,\ldots,x_{2k-1},x_{2k-1},\ldots,x_1\rfloor = \lfloor x_1,\ldots,x_{2k-1}\rfloor = m\,.$$

From (9.1.5) follows that $_{\lfloor x_1,\ldots,x_{2k-1}\rfloor}\pi_{\lfloor x_{2k-1},\ldots,x_1\rfloor}$ is in $L\circledast L$. Q.E.D.

(9.4) Lemma Let L be a meridian libra and let $\{s,x,y\}\subset L$ be such that

$$_s\pi_s(y) = x\,. \tag{1}$$

Then there exists $\{r,t\}\subset L$ such that

$$\lfloor\!\lceil [r,r],[s,s],[t,t] \rceil\!\rfloor = [x,y]\,. \tag{2}$$

Proof. By (8.33) there exists

$$\{t,r\}\subset s^{\diamond} \tag{3}$$

such that

$$\lfloor t,s,r\rfloor = y\,. \tag{4}$$

Then

$$x \overset{\text{by (1)}}{=\!=\!=} \lfloor s,y,s\rfloor \overset{\text{by (4)}}{=\!=\!=} \lfloor s,\lfloor t,s,r\rfloor,s\rfloor = \lfloor s,r,s,t,s\rfloor \overset{\text{by (3)}}{=\!=\!=} \lfloor s,s,r,s,r\rfloor = \lfloor r,s,t\rfloor\,.$$

Consequently

$$[x,y] = [\,\lfloor r,s,t\rfloor,\lfloor t,s,r\rfloor\,] = \lfloor\!\lceil [r,r],[s,s],[t,t] \rceil\!\rfloor\,.$$

Q.E.D.

(9.5) Theorem Let L be a meridian libra. Then

$$L\boxtimes L = L\times L \iff (\forall\, \{x,y\}\in L)(\exists\, \phi\in L\circledast L)\quad \phi(x) = y\,. \tag{1}$$

Proof. In view of (9.3) we need only show that $L\boxtimes L = L\times L$ if the right-hand side of (1) holds. We shall presume then that the right-hand side of (1) holds and that $\{m,n\}\subset L$ — and then proceed to show that $[m,n]$ is in $L\boxtimes L$. Towards this end we select ϕ as in (9.3.1) and then apply (9.1.4) to obtain $k\in\mathbb{N}$ and $x\in L^{\underline{2k-1}}$ such that

$$n = {_{x_1}\pi_{x_1}}\circ\ldots\circ{_{x_{2k-1}}\pi_{x_{2k-1}}}(m) \implies m = {_{x_{2k-1}}\pi_{x_{2k-1}}}\circ\ldots\circ{_{x_1}\pi_{x_1}}(n)\,. \tag{2}$$

We define $y_0 \equiv n$, and for each $i \in \underline{2k\text{-}1}$, we shall abbreviate ${}_{x_i}\pi_{x_i} \circ \ldots \circ {}_{x_{2k-1}}\pi_{x_{2k-1}}(m)$ to y_i. From Lemma (9.4) follows that

$$(\forall\, i \in \underline{2k\text{-}1}) \quad [y_i, y_{i-1}] \in L \boxtimes L\,. \tag{3}$$

It follows from (9.1.5) that

$$(\forall\, i \in \underline{2k\text{-}1}) \quad [y_{i-1}{}^{\varsigma}, y_i] \in L \boxtimes L\,. \tag{4}$$

Since $L \boxtimes L$ is balance, it follows that

$$\left[\!\left[[y_0,y_1],[y_2,y_1],[y_2,y_3],\ldots,[y_{2k-2},y_{2k-3}],[y_{2k-2},y_{2k-1}] \right]\!\right]$$

is in $L \boxtimes L$. Furthermore

$$\left[\!\left[[y_0,y_1],[y_2,y_1],[y_2,y_3],\ldots,[y_{2k-2},y_{2k-3}],[y_{2k-2},y_{2k-1}] \right]\!\right] \overset{\text{by (9.1.2)}}{=\!=}$$

$$\left[\lfloor y_0,y_2,y_2 \ldots, y_{2k-2},y_{2k-2} \rfloor, \lfloor y_1,y_1,y_3,\ldots,y_{2k-3},y_{2k-3},y_{2k-1} \rfloor \right] = [y_0,y_{2k-1}] = [m,n]$$

and so $[n,m]$ is in $L \boxtimes L$. From (9.1.5) follows that $[m,n]$ is in $L \boxtimes L$. Q.E.D.

(9.6) Theorem Let $\mathbb{L}$ be as in Section (7) and let U and V be the two connected components, or halves, of $\mathbb{L}$, as in (7.3). Let a be an element of $\mathbb{L}$. Then

$$(\forall\, x \in \mathbb{L}) \quad {}_a\pi_a(x) \in U \iff x \in U \quad \text{and} \quad {}_a\pi_a(x) \in V \iff x \in V\,. \tag{1}$$

Proof. Without loss of generality we may presume that a is in U. Suppose that x is in U. If $x = a$, then ${}_a\pi_a(x) = x$ and so (1) holds trivially. Suppose then that $a \neq x$. Let S be be closed line segment which lies in U with endpoints x and a. Since ${}_a\pi_a$ is continuous, $\overrightarrow{{}_a\pi_a}(S)$ is a closed line segment. Since $a = {}_a\pi_a(a)$ is in $\overrightarrow{{}_a\pi_a}(S)$, and since a is in U, it follows that $\overrightarrow{{}_a\pi_a}(S) \subset U$ and so, in particular, (1) holds.

We have effectively shown that $\overrightarrow{{}_a\pi_a}(U) = U$. Since ${}_a\pi_a$ is an involution, it follows that $\overrightarrow{{}_a\pi_a}(V) = V$ as well. Q.E.D.

(9.7) Example: the Circle or Line Meridian We consider the meridian libra of Section (7), which is isomorphic with the libra of homographies of the circle meridian (and the line meridian).

Recall that the toroid T separates its complement $\mathbb{L}$ in $\mathbb{E}$ into two connected components or halves. Let m be in one half and n be in another. If $L \boxtimes L$ were the same as $L \times L$, then it would follow from Theorem (9.3) that there would be some $\phi \in L \circledast L$ such that $\phi(m) = n$. But ϕ being a composition of elements ${}_a\pi_a$, each of which by Theorem (9.6) would send m to the same half in which it resides, ϕ could not send m to n. It follows that, for the meridian libra of the circular meridian

$$L \times L \neq L \boxtimes L\,. \tag{1}$$

We shall see *infra* that the balanced set $L \boxtimes L$ has exactly one coset in $L \times L$.

(9.8) Example: the Sphere Meridian Let S be a sphere in euclidian space **E** as in (1.11). Let L be the libra of holomorphies of S. Let Let m and n be distinct elements of L. Then $\overleftrightarrow{m,n}$ is either incident or accident, there being no excident lines in this example.

Suppose that $\overleftrightarrow{m,n}$ is incident. Then there are two elements 0 and ∞ of M which are fixed by all members of $\overleftrightarrow{m,n}$. Letting 1 in M be distinct from 0 and ∞, we have a basis for a field F. Let $r \equiv m(1)$ and $s \equiv n(1)$. Then

$$m = \begin{pmatrix} 0 & r \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad n = \begin{pmatrix} 0 & s \\ 1 & 0 \end{pmatrix}.$$

Let $w \in F$ be such that $w^2 = rs$. Then

$$\left\lfloor \begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix}, n, \begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix} \right\rfloor = \begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & s \\ 1 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & w^2 \\ s & 0 \end{pmatrix} = \begin{pmatrix} 0 & r \\ 1 & 0 \end{pmatrix} = m. \tag{1}$$

Suppose now that $\overleftrightarrow{m,n}$ is accident. There there is exactly one element ∞ fixed by all elements of $\overleftrightarrow{m,n}$. This time we shall let 0 be any element of M distinct from ∞, let $1 \equiv m(0)$ and let $s \equiv n(0)$. Then

$$m = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad n = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

For $a \equiv \begin{pmatrix} 1 & \frac{1-s}{2} \\ 0 & 1 \end{pmatrix}$,

$$\lfloor a,n,a \rfloor = \begin{pmatrix} 1 & \frac{1-s}{2} \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & \frac{1-s}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = m. \tag{2}$$

From (1) and (2) follows that $\lfloor a,n,a \rfloor = m$. Theorem (9.5) now implies that

$$L \boxtimes L = L \times L. \tag{3}$$

(9.9) Notation and Definitions Let $\mathcal{T}$ be a cartesian aggregate of a libra L, and let ρ be any representation of L equivalent to the left $\mathcal{T}$-inner representation. For $b \in \boxed{\rho}$ and $a \in \boxed{\rho}$ we shall write

$$\boxed{\rho}_b \equiv \{[x,b] : x \in \boxed{\rho}\} \quad \text{and} \quad \boxed{\rho}_a \equiv \{[a,y] : y \in \boxed{\rho}\} \tag{1}$$

and say that $\boxed{\rho}_b$ is a **left cross section** and that $\boxed{\rho}_a$ is a **right cross section**.

A permutation of $\boxed{\rho} \times \boxed{\rho}$ which sends left cross sections to right cross sections and which on each left cross section is equal to some ρ_x on that cross section, will be said to be **left ρ-contrajective**. A permutation of $\boxed{\rho} \times \boxed{\rho}$ which sends right cross sections to left cross sections and which on each right cross section is equal to some ρ_x on that cross section, will be said to be **right ρ-contrajective**. A permutation which is both left and right ρ-contrajective is said to be **ρ-contrajective**.

A permutation of $\boxed{\rho} \times \boxed{\rho}$ which sends left cross sections to left cross sections, and is such that for each left cross section, there exists $\{u,v\} \subset L$ it equals $\rho_u{}^{-1} \circ \rho_v$ on that cross section, will be said to be **left ρ-cojective**. A permutation of $\boxed{\rho} \times \boxed{\rho}$ which sends right cross sections to right cross sections, and is such that for each right cross section, there exists $\{u,v\} \subset L$ it equals $\rho_u \circ \rho_v{}^{-1}$ on that cross section, will be said to be **right ρ-cojective**. Permutations which are both left and right ρ-cojective will be said to be **ρ-cojective**.

(9.10) Theorem Let $\mathcal{T}$ be a cartesian aggregate of a libra L, and let ρ be any representation of L equivalent to the left $\mathcal{T}$-inner representation. Let ϕ be a ρ-contrajective permutation of $\boxed{\rho} \times \boxed{\rho}$. Then

$$(\exists \{r,s\} \subset L) \quad \overleftrightarrow{\rho}_{\lfloor r,s \rfloor} = \phi. \tag{1}$$

Proof. Define $f \big| \boxed{\rho} \hookrightarrow \boxed{\rho}$ and $g \big| \boxed{\rho} \hookrightarrow \boxed{\rho}$ by

$$(\forall\, x \in \boxed{\rho})\quad \vec{\phi}\,(\boxed{\rho}_{x}) = \boxed{\rho}_{f(x)} \quad \text{and} \quad (\forall\, y \in \boxed{\rho})\quad \vec{\phi}\,(\boxed{\rho}_{x}) = \boxed{\rho}_{g(y)}\,. \tag{2}$$

Define $\theta \big| \boxed{\rho} \times \boxed{\rho} \hookrightarrow \boxed{\rho} \times \boxed{\rho}$ by

$$(\forall\, [x,y] \in \boxed{\rho} \times \boxed{\rho})\quad \{\theta([x,y])\} \equiv \boxed{\rho}_{f(x)} \cap \boxed{\rho}_{g(y)}\,. \tag{3}$$

For $[x,y] \in \boxed{\rho} \times \boxed{\rho}$

$$\{\phi([x,y])\} = \vec{\phi}\,(\boxed{\rho}_{x}) \cap \vec{\phi}\,(\boxed{\rho}_{y}) \xlongequal{\text{by (2) and (3)}} \{\theta([x,y])\}\,. \tag{4}$$

For $b \in \boxed{\rho}$, our hypothesis provides $\{r,s\} \subset L$ such that ϕ equals $\overleftrightarrow{\rho}_{[r,s]}$ on $\boxed{\rho}_{b}$: to wit

$$(\forall\, x \in \boxed{\rho})\quad \phi([x,b]) = \overleftrightarrow{\rho}_{[r,s]}([x,b]) = [\rho_s(b), \rho_r(x)] \xRightarrow{\text{by (2), (3) and (4)}}$$

$$\{[\rho_s(b), \rho_r(x)]\} = \boxed{\rho}_{f(x)} \cap \boxed{\rho}_{g(b)} = \{[g(b), f(x)]\} \Longrightarrow$$

$$[\rho_s(b), \rho_r(x)] = [g(b), f(x)]\,. \tag{5}$$

A priori the heritage of s and r depended on $b \in \boxed{\rho}$, but it follows from (5) that $\rho_r(x)$ does not. Since ρ is faithful, it follows that r does not. By an argument symmetric and analagous to the preceding, we see that s is completly determined as well. Thus (1) holds. Q.E.D.

(9.11) Corollary Let $\mathcal{T}$ be a cartesian aggregate of a libra L , and let ρ be any representation of L equivalent to the left $\mathcal{T}$-inner representation. Let γ be any ρ-cojective mapping. Then

$$(\exists\, \{r,s,u,v\} \subset L)\quad \overleftrightarrow{\rho}_{\lfloor r,s \rfloor} \circ \overleftrightarrow{\rho}_{\lfloor u,v \rfloor} = \gamma\,. \tag{1}$$

Proof. Let $\{u,v\} \subset L$ and define $\phi \equiv \gamma \circ (\overleftrightarrow{\rho}_{\lfloor u,v \rfloor})^{-1}$. Then ϕ is ρ-controvariant and so Theorem (9.6) implies that there exists $\{r,s\} \subset L$ such that (9.10.1) holds. It follows that (1) holds. Q.E.D.

(9.12) Lemma Let $\mathcal{T}$ be a meridian aggregate of a libra L , and let ρ be any representation of L equivalent to the left $\mathcal{T}$-inner representation. Let ϕ be in $\mathcal{M}$, where $\mathcal{M}$ is as in (8.10). Then

$$(\forall\, a \in L)(\exists\, z \in a^{\blacklozenge})\quad \phi = {\rho_a}^{-1} \circ \rho_z\,. \tag{1}$$

Proof. Let a be an element of L. By the definition of the meridian structure on M, we have

$$\mathcal{G} = \{{\rho_a}^{-1} \circ \rho_z : z \in L\}\,.$$

Thus $\phi = {\rho_a}^{-1} \circ \rho_z$ for some $z \in L$. Since ϕ is an involution,

$${\rho_a}^{-1} \circ \rho_z = ({\rho_a}^{-1} \circ \rho_z)^{-1} = {\rho_z}^{-1} \circ \rho_a \Longrightarrow \rho_z = \rho_a \circ {\rho_z}^{-1} \circ \rho_a\,.$$

Q.E.D.

(9.13) Theorem Let $\mathcal{T}$ be a meridian aggregate of a properly linear libra L , and let ρ be any representation of L equivalent to the left $\mathcal{T}$-inner representation. Let γ be a ρ-cojective mapping and let a be an element of L . then

$$(\exists\,\{b,c,d,e\}\subset L)\quad a=\lfloor b,a,b\rfloor=\lfloor c,a,c\rfloor=\lfloor d,a,d\rfloor=\lfloor e,a,e\rfloor \quad\text{and}\quad \overleftrightarrow{\rho}_{\lfloor a,a\rfloor}\circ\overleftrightarrow{\rho}_{\lfloor\lfloor e,a,c\rfloor,\lfloor d,a,b\rfloor\rfloor}=\gamma\,. \tag{1}$$

Proof. By (9.11) there exists $\{r,s,u,v\}\subset L$ such that

$$\gamma=\overleftrightarrow{\rho}_{\lfloor r,s\rfloor}\circ\overleftrightarrow{\rho}_{\lfloor u,v\rfloor}\,. \tag{2}$$

The mapping ${\rho_s}^{-1}\circ\rho_u$ is in $\mathcal{G}$ and so by (1.3), ${\rho_s}^{-1}\circ\rho_u$ is either the identity function, an involution, or a composition of involutions. In the first case we set $e\equiv a$ and $c\equiv a$. In the second case we set $e\equiv a$ and $c\equiv\lfloor a,s,u\rfloor$. In the third case we apply Lemma (9.12) to obtain $\{c,e\}\subset a^{\diamond}$ such that $\rho_s={\rho_a}^{-1}\circ\rho_e$ and $\rho_u={\rho_a}^{-1}\circ\rho_c$. For each of these cases $\rho_s={\rho_e}^{-1}\circ\rho_a$, which implies

$${\rho_s}^{-1}\circ\rho_u={\rho_a}^{-1}\circ\rho_e\circ{\rho_a}^{-1}\circ\rho_c={\rho_a}^{-1}\circ\rho_{\lfloor e,a,c\rfloor}\,.$$

Reasoning similarly, we can find $\{b,d\}\subset L$ such that

$$a=\lfloor b,a,b\rfloor=\lfloor d,a,d\rfloor \quad\text{and}\quad \rho_r\circ{\rho_v}^{-1}=\rho_a\circ{\rho_{\lfloor d,a,b\rfloor}}^{-1}\,.$$

For any $[x,y]\in\boxed{\rho}\times\boxed{\rho}$

$$\overleftrightarrow{\rho}_{\lfloor a,a\rfloor}\circ\overleftrightarrow{\rho}_{\lfloor\lfloor e,a,c\rfloor,\lfloor d,a,b\rfloor\rfloor}([x,y])=[{\rho_a}^{-1}\circ\rho_{\lfloor e,a,c\rfloor}(x)\,,\rho_a\circ{\rho_{\lfloor d,a,d\rfloor}}^{-1}(y)\,]=$$
$$[{\rho_s}^{-1}\circ\rho_u(x)\,,\rho_r\circ\rho_v(y)^{-1}\,]=\overleftrightarrow{\rho}_{\lfloor r,s\rfloor}\circ\overleftrightarrow{\rho}_{\lfloor u,v\rfloor}([x,y])$$

which with (2) implies (1). Q.E.D.

(9.14) Corollary Let $\mathcal{T}$ be a meridian aggregate of a properly linear libra L, and let ρ be any representation of L equivalent to the left $\mathcal{T}$-inner representation. Let ψ be a ρ-cojective mapping and let a be an element of L. Then

$$(\exists\,\{b,c,d,e\}\subset L)\quad a=\lfloor b,a,b\rfloor=\lfloor c,a,c\rfloor=\lfloor d,a,d\rfloor=\lfloor e,a,e\rfloor \quad\text{and}\quad \overleftrightarrow{\rho}_{\lfloor\lfloor d,a,b\rfloor,\lfloor e,a,c\rfloor\rfloor}\circ\overleftrightarrow{\rho}_{\lfloor a,a\rfloor}=\psi\,. \tag{1}$$

Proof. Let $\psi\equiv\gamma^{-1}$ in Theorem (9.13) and observe that

$$\overleftrightarrow{\rho}_{\lfloor\lfloor d,a,b\rfloor,\lfloor e,a,c\rfloor\rfloor}\circ\overleftrightarrow{\rho}_{\lfloor a,a\rfloor}\text{ is the inverse of }\overleftrightarrow{\rho}_{\lfloor a,a\rfloor}\circ\overleftrightarrow{\rho}_{\lfloor\lfloor e,a,c\rfloor,\lfloor d,a,b\rfloor\rfloor}\,.$$

Q.E.D.

(9.15) Corollary Let L be any meridian libra with meridian aggregate $\mathcal{T}$. Let a be any element of L. Then[19]

$$\mathfrak{Group}(\mathcal{T})=\{\lfloor u,a\circledcirc a,v\rfloor:\{u,v\}\subset L\}\,. \tag{1}$$

Proof. By definition,

$$\mathfrak{Group}(\mathcal{T})=\{(t\circledcirc m)\circ(n\circledcirc w):\{t,m,n,w\}\subset L\}\,. \tag{2}$$

In view of (9.14) there exists $\{u,v\}\subset L$ such that $(t\circledcirc m)\circ(n\circledcirc w)=(u\circledcirc v)\circ(a\circledcirc a)$. From (2) follows

$$\mathfrak{Group}(\mathcal{T})=\{(u\circledcirc v)\circ(a\circledcirc a):\{u,v\}\subset L\}\overset{\text{by (5.7.2)}}{=\!=\!=}\{\lfloor u,a\circledcirc a,u\rfloor:\{u,v\}\subset L\}\,.$$

Q.E.D.

19 *Cf.* (5.7).

(9.16) Example: The Toroid Let T be a toroid as in Section (7). The set T is a disjoint union of lines or **rules** in two ways. These two pairwise disjoint families of rules are called reguli, which we denote [T] and [T]. They endow T with a sort of geometry in the sense of Klein's Erlanger Programm. One can consider the group $\mathfrak{G}$ of all permutations ϕ of T which send rules to rules and such that restricted to any rule, are meridian isomorphisms. Since the rules in each regulus are pairwise disjoint, and rules in different reguli intersect in a single point, such a permutation ϕ must sent all the rules in one regulus [T] either onto all the rules of [T] , or onto all the rules of [T] : in the first case we shall say the permutation of T is **cojective** and, in the second case, **contrajective**.

The functions x^{τ}, for $x \in \mathbb{L}$, which permute T by drawing lines through x which intersect T, and interchanging the two elements of T on such lines, are clearly examples of contrajective permutations of T. This begs the question: what exactly are the elements of $\mathfrak{G}$? We are in a position to answer that question:

It follows from (9.6) that

$$\text{the contrajective permutations of T are precisely the mappings } \delta_{[a,b]} \text{ for } \{a,b,c\} \subset \mathbb{L} \tag{1}$$

as defined in (7.9.3). It follows from (9.9) that

$$\text{the cojective permutations of T are precisely the mappings } \delta_{[a,b]} \circ c^{\tau} \text{ for } \{a,b,c\} \subset \mathbb{L}\,. \tag{2}$$

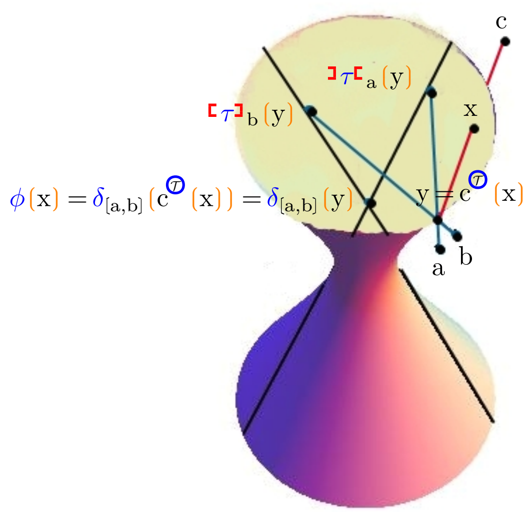

Fig. 19: General Cojective Permutation ϕ of T at a point x of T

The identity mapping on T is clearly cojective. The group $\mathfrak{G}$ thus has a subgroup consisting of the covariant permutations of T, and this subgroup has only one coset: the family of contravariant mappings.

10. Notation

11. Index

12. Acknowledgement

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