Given n numbers $a_1, a_2, \ldots, a_n$, calculate $\sum_{i<j} (a_j - a_i)^2$ in linear time.

Ans:

$$\begin{aligned}\sum_{i<j} (a_j - a_i)^2 &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (a_j - a_i)^2 \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (a_j^2 - 2a_j a_i + a_i^2) \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_j^2 - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 2a_j a_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_i^2\end{aligned}$$

Calculate the three terms in linear time then we are done.

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_i^2 = \sum_{i=1}^{n-1} (n-i) a_i^2 \quad (1)$$

$A_1 = 0$
for $i = n-1$ **downto 1 do**
  $A_1 += (n-i) a_i^2$
end

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 2a_j a_i = \sum_{i=n-1}^{1} 2a_i \sum_{j=n}^{i+1} a_j \quad (2)$$

$A_2 = 0; tmp = 0$
for $i = n-1$ **downto 1 do**
  $tmp += a_{i+1}$
  $A_2 += 2a_i \cdot tmp$
end

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_j^2 \quad (3)$$

$A_3 = 0; tmp = 0$
for $i = n-1$ **downto 1 do**
  $tmp += a_{i+1}^2$
  $A_3 += tmp$
end
$Sum = A_1 - A_2 + A_3$