Theory of Computation

Course note based on *Computability, Complexity, and Languages: Fundamentals of Theoretical Computer Science*, 2nd edition, authored by Martin Davis, Ron Sigal, and Elaine J. Weyuker.

course note prepared by

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About This Course Note

- It is prepared for the course *Theory of Computation* taught at the National Taiwan University in Spring 2010.
- It follows very closely the book *Computability, Complexity, and Languages: Fundamentals of Theoretical Computer Science*, 2nd edition, by Martin Davis, Ron Sigal, and Elaine J. Weyuker. Morgan Kaufmann Publishers. ISBN: 0-12-206382-1.
- It is available from Tyng-Ruey Chuang's web site:

  `http://www.iis.sinica.edu.tw/~trc/`

This course aims to cover ...

- the development of computability theory using an extremely simple abstract programming language,
- the various different formulations of computability and their equivalence,
- the expressiveness and limitation of various kinds of automata and formal languages, and
- the basics of the theory of computational complexity.

By the end of this course, you should be able to ...

- appreciate the existence of universal digital computers,
- understand there are well-defined functions that cannot be computed even by the universal computers,
- know that certain problems are truly harder than others,
- use various formalized computation models to solve your problems, and
- show that some problems are just too difficult for the models at hand.

Textbook

Martin Davis, Ron Sigal, and Elaine J. Weyuker. *Computability, Complexity, and Languages: Fundamentals of Theoretical Computer Science*, 2nd edition. February 1994, Morgan Kaufmann. ISBN: 0122063821.

- Written for people who may know programming, but from a mathematical view of the subjects. Enjoyably readable but very rigorous.
- "It is our purpose ... to provide an introduction to the various aspects of theoretical computer science for undergraduate and graduate students that is sufficiently comprehensive that ... research papers will become accessible to our readers." (the authors)
- We will cover just one half of the materials in the book.

Schedule (1/2)

02/24 Preliminaries; A Programming Language. (1; 2.1–2.2)

03/03 Computable Functions; Primitive Recursive Functions. (2.3–2.5; 3.1–3.4)

03/10 Coding Programs by Numbers. (3.5–3.8; 4.1)

03/17 The Halting Problem; Universality. (4.2–4.3)

03/24 Recursively Enumerable Sets. (4.4–4.5)

03/31 Diagonalization and Reducibility. (4.6–4.8)

04/07 *(no class on 04/07)*

04/14 mid-term examination

04/21 A Computable Function That Is Not Primitive Recursive. (4.9)

Schedule (2/2)

04/28 Regular Languages, Part 1. (9.1–9.4)

04/05 Regular Languages, Part 2. (9.5–9.7)

05/12 Context-Free Languages, Part 1. (10.1–10.4)

05/19 Context-Free Languages, Part 2. (10.5–10.9)

05/26 Calculations on Strings (5.1–5.6)

06/02 Turing Machines (6.1–6.5)

06/09 Abstract Complexity (14.1–14.4)

06/16 *(no class on 06/16)*

06/23 final examination

Outline of Today's Lecture

- Review some preliminary materials.
- Define an abstract programming language $\mathscr{S}$ that is extremely simple.
- Write some programs in $\mathscr{S}$.

1 Preliminaries (1)

1.1 Sets, n-tuples, and functions (1.1, 1.2)

Cartesian Product

- If $S_1, S_2, \ldots, S_n$ are given sets, then we write $S_1 \times S_2, \times \cdots \times S_n$ for the set of all n-tuples $(a_1, a_2, \ldots, a_n)$ such that $a_1 \in S_1, a_2 \in S_2, \ldots, a_n \in S_n$.
- $S_1 \times S_2, \times \cdots \times S_n$ is called the *Cartesian product* of $S_1, S_2, \ldots, S_n$.
- In case $S_1 = S_2 = \cdots = S_n = S$ we write S^n for the Cartesian product $S_1 \times S_2, \times \cdots \times S_n$.

Functions

- A function f is a set whose members are ordered pairs (i.e., 2-tuples) and has the special property
$$(a, b) \in f \text{ and } (a, c) \in f \quad \text{implies} \quad b = c.$$
We write $f(a) = b$ to mean that $(a, b) \in f$.
- The set of all a such that $(a, b) \in f$ for some b is called the *domain* of f. The set of all $f(a)$ for a in the domain of f is called the *range* of f.
- A *partial function* on a set S is a function whose domain is a subset of S. If a partial function on S has the domain S, then it is called a *total function*.
- We write $f(a) \downarrow$ and say that $f(a)$ is *defined* if a is in the domain of f; if a is not in the domain of f, we write $f(a) \uparrow$ and say that $f(a)$ is *undefined*.

Examples of Functions

- Let f be the set of ordered pairs (n, n^2) for $n \in N$. Then, for each $n \in N$, $f(n) = n^2$. The domain of f is N. The range of f is the set of perfect squares. f is a total function.
- Assuming N is our universe, an example of a partial function on N is given by $g(n) = \sqrt{n}$. The domain of g is the set of perfect squares. The range of g is N. g is not a total function.
- For a partial function f on a Cartesian product $S_1 \times S_2, \times \cdots \times S_n$, we write $f(a_1, \ldots, a_n)$ rather than $f((a_1, \ldots, a_n))$.
- A partial function f on a set S^n is called an *n-ary* partial function on S, or a function of n variables on S. We use *unary* and *binary* for 1-ary and 2-ary, respectively.

1.2 Predicates (1.4)

Predicate

A *predicate*, or a *Boolean-valued function*, on a set S is a *total* function P on S such that for each $a \in S$, either
$$P(a) = \text{TRUE} \quad \text{or} \quad P(a) = \text{FALSE}$$
We also identify the truth value TRUE with number 1 and the truth value FALSE with number 0.

Logic Connectives

The three *logic connectives*, or *propositional connectives*, $\sim, \vee, \&$ are defined by the two tables below.

p	$\sim p$
0	1
1	0

p	q	$p \,\&\, q$	$p \vee q$
1	1	1	1
0	1	0	1
1	0	0	1
0	0	0	0

Characteristic Function

Given a predicate P on a set S, there is a corresponding subset R of S consisting of all elements $a \in S$ for which $P(a) = 1$. We write

$$R = \{a \in S \mid P(a)\}.$$

Conversely, given a subset R of a given set S, the expression $x \in R$ defines a predicate P on S:

$$P(x) = \begin{cases} 1 & \text{if } x \in R \\ 0 & \text{if } x \notin R. \end{cases}$$

The predicate P is called the *characteristic function* of the set R. Note the easy translations between the two notations:

$$\begin{aligned} \{x \in S \mid P(x) \,\&\, Q(x)\} &= \{x \in S \mid P(x)\} \cap \{x \in S \mid Q(x)\}, \\ \{x \in S \mid P(x) \vee Q(x)\} &= \{x \in S \mid P(x)\} \cup \{x \in S \mid Q(x)\}, \\ \{x \in S \mid \sim P(x)\} &= S - \{x \in S \mid P(x)\}. \end{aligned}$$

1.3 Quantifiers (1.5)

Bounded Existential Quantifier

Let $P(t, x_1, \ldots, x_n)$ be a $(n+1)$-ary predicate. Let predicate $Q(y, x_1, \ldots, x_n)$ be defined by

$$\begin{aligned} Q(y, x_1, \ldots, x_n) &= P(0, x_1, \ldots, x_n) \\ &\vee \; P(1, x_1, \ldots, x_n) \\ &\vee \; \ldots \\ &\vee \; P(y, x_1, \ldots, x_n) \end{aligned}$$

That is, $Q(y, x_1, \ldots, x_n)$ is true if there is a value $t \leq y$ such that $P(t, x_1, \ldots, x_n)$ is true. We write this predicate Q as

$$(\exists t)_{\leq y} P(t, x_1, \ldots, x_n)$$

"$(\exists t)_{\leq y}$" is called a *bounded existential quantifier.*

Bounded Universal Quantifier

Let $P(t, x_1, \ldots, x_n)$ be a $(n+1)$-ary predicate. Let predicate $Q(y, x_1, \ldots, x_n)$ be defined by

$$\begin{aligned} Q(y, x_1, \ldots, x_n) &= P(0, x_1, \ldots, x_n) \\ &\& \quad P(1, x_1, \ldots, x_n) \\ &\& \quad \ldots \\ &\& \quad P(y, x_1, \ldots, x_n) \end{aligned}$$

That is, $Q(y, x_1, \ldots, x_n)$ is true if for all value $t \leq y$ such that $P(t, x_1, \ldots, x_n)$ is true. We write this predicate Q as

$$(\forall t)_{\leq y} P(t, x_1, \ldots, x_n)$$

"$(\forall t)_{\leq y}$" is called a *bounded universal quantifier*.

1.4 Proofs (1.6, 1.7)

Proof by Contradiction

In a *proof by contradiction*, we begin by assuming the assertion we wish to prove is *false*. We then derive a contradiction based on this (faulty) assumption along with (faultless) logical reasoning. We then conclude that the original assertion must be true.

Proof by Contradiction: Example

Prove that the equation $2 = (m/n)^2$ has no solution $m, n \in N$. *Proof.* Assume $2 = (m/n)^2$ *has* a solution $m, n \in N$. Then it must also have a solution where not both m and n are even. This is so because we can repeatedly "cancel" 2 from m and n until at least one of them becomes *odd*, and still have the two "reduced" numbers as a solution. However, the equation $2 = (m/n)^2$ can be rewritten as $m^2 = 2n^2$ which shows that m must be even. Let $m = 2k$, then $m^2 = (2k)^2 = 4k^2$. But this implies $n^2 = 2k^2$. Thus n is even. Now both m and n are even, which is a contradiction. We conclude that $2 = (m/n)^2$ has no solution $m, n \in N$. □

Mathematical Induction

Given a predicate $P(x)$, and the assertion "$P(n)$ is true for all $n \in N$", we can use mathematical induction to try to establish this assertion. One proceeds by proving a pair of auxiliary statements about $P(x)$, namely,

$P(0)$

and

For all $n \in N$, $P(n)$ implies $P(n+1)$

In the second statement above, $P(n)$ is called an induction hypothesis. If both statements above are proved to be true, one then concludes that

For all $n \in N$, $P(n)$

Mathematical Induction: Example

Prove that for all $n \in N, \sum_{i=0}^{n}(2i+1) = (n+1)^2$. *Proof.* For $n = 0$, then $\sum_{i=0}^{0}(2i+1) = 1 = (0+1)^2$, which is true. It remains to show that for all $n \in N$, if $\sum_{i=0}^{n}(2i+1) = (n+1)^2$ is true, then $\sum_{i=0}^{n+1}(2i+1) = (n+2)^2$ is also true. We expand $\sum_{i=0}^{n+1}(2i+1)$ by its definition,

$$\begin{aligned}\sum_{i=0}^{n+1}(2i+1) &= \sum_{i=0}^{n}(2i+1) + 2(n+1) + 1 \\ &= (n+1)^2 + 2(n+1) + 1 \qquad \text{(by induction hypothesis)} \\ &= (n+2)^2.\end{aligned}$$

We conclude that for all $n \in N, \sum_{i=0}^{n}(2i+1) = (n+1)^2$. □

2 Programs and Computable Functions (2)

2.1 A Programming Language (2.1)

The Programming Language $\mathscr{S}$

- Values: natural numbers only, but of unlimited precision.
- Variables:
  - Input variables $X_1, X_1, X_3, \ldots$
  - An output variable Y
  - Local variables $Z_1, Z_1, Z_3, \ldots$
- Instructions:

  $V \leftarrow V + 1$ Increase by 1 the value of the variable V.

  $V \leftarrow V - 1$ If the value of V is 0, leave it unchanged; otherwise decrease by 1 the value of V.

  IF $V \neq 0$ **GOTO** L If the value of V is nonzero, perform the instruction with label L next; otherwise proceed to the next instruction in the list.
- Labels: $A_1, B_1, C_1, D_1, E_1, A_2, B_2, C_2, D_2, E_2, A_3, \ldots$
- Exit label: E.
- All variables and labels are in the global scope.

2.2 Some Examples of Programs (2.2)

Programming in $\mathscr{S}$

- A program is a list (i.e., a finite sequence) of instructions.
- *The output variable Y and the local variables Z_i initially have the value* 0.
- A program halts when there is no more instruction to execute.
- A program also halts if an instruction labeled L is to be executed, but there is no instruction with that label.
- What does this program do?

$$\begin{array}{ll} [A] & X \leftarrow X - 1 \\ & Y \leftarrow Y + 1 \\ & \text{IF } X \neq 0 \text{ GOTO } A \end{array}$$

A Bug?

- What does this program do?

$$\begin{array}{ll} [A] & X \leftarrow X - 1 \\ & Y \leftarrow Y + 1 \\ & \text{IF } X \neq 0 \text{ GOTO } A \end{array}$$

- The above program *computes* the function

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ x & \text{otherwise.} \end{cases}$$

A Program That Computes $f(x) = x$

$$\begin{array}{ll} [A] & \text{IF } X \neq 0 \text{ GOTO } B \\ & Z \leftarrow Z + 1 \\ & \text{IF } Z \neq 0 \text{ GOTO } E \\ [B] & X \leftarrow X - 1 \\ & Y \leftarrow Y + 1 \\ & Z \leftarrow Z + 1 \\ & \text{IF } Z \neq 0 \text{ GOTO } A \end{array}$$

- What does Z actually do?
- What does the following do?

$Z \leftarrow Z + 1$
IF $Z \neq 0$ GOTO L

- That is an unconditional goto!

  GOTO L

A *Macro* for Unconditional GOTO

- Before macro expansion:

  $[A]$ IF $X \neq 0$ GOTO B
  GOTO E
  $[B]$ $X \leftarrow X - 1$
  $Y \leftarrow Y + 1$
  GOTO A

- After macro expansion:

  $[A]$ IF $X \neq 0$ GOTO B
  $Z_1 \leftarrow Z_1 + 1$
  IF $Z_1 \neq 0$ GOTO E
  $[B]$ $X \leftarrow X - 1$
  $Y \leftarrow Y + 1$
  $Z_2 \leftarrow Z_2 + 1$
  IF $Z_2 \neq 0$ GOTO A

- *Fresh local variables are always used during macro expansions.*

Copy The Value of Variable X to Variable Y

- $[A]$ IF $X \neq 0$ GOTO B
  GOTO E
  $[B]$ $X \leftarrow X - 1$
  $Y \leftarrow Y + 1$
  GOTO A

- Anything wrong?

- The value of X is "destroyed" while copied to Y!

Copy The Value of Variable X to Variable Y, Continued

- $[A]$ IF $X \neq 0$ GOTO B
  GOTO C
  $[B]$ $X \leftarrow X - 1$
  $Y \leftarrow Y + 1$
  $Z \leftarrow Z + 1$
  GOTO A
  $[C]$ IF $Z \neq 0$ GOTO D
  GOTO E
  $[D]$ $Z \leftarrow Z - 1$
  $X \leftarrow X + 1$
  GOTO C
- Anything wrong?
- This program is correct only when Y and Z are initialized to the value 0. It cannot be used as a macro.

A Macro for $V \leftarrow V'$

- $V \leftarrow 0$
  $[A]$ IF $V' \neq 0$ GOTO B
  GOTO C
  $[B]$ $V \leftarrow V' - 1$
  $V \leftarrow V + 1$
  $Z \leftarrow Z + 1$
  GOTO A
  $[C]$ IF $Z \neq 0$ GOTO D
  GOTO E
  $[D]$ $Z \leftarrow Z - 1$
  $V' \leftarrow V' + 1$
  GOTO C
- Anything wrong?
- $V \leftarrow 0$ is not an instruction in $\mathscr{S}$.

A Macro for $V \leftarrow 0$

$[L]$ $V \leftarrow V - 1$
IF $V \neq 0$ GOTO L

A Program That Computes $f(x_1, x_2) = x_1 + x_2$

$Y \leftarrow X_1$
$Z \leftarrow X_2$
$[B]$ IF $Z \neq 0$ GOTO A
GOTO E
$[A]$ $Z \leftarrow Z - 1$
$Y \leftarrow Y + 1$
GOTO B

Note that Z is used to preserve the value of X_2 so that it will not be destroyed during the computation.

A Program That Computes $f(x_1, x_2) = x_1 \cdot x_2$

- $Z_2 \leftarrow X_2$
  $[B]$ IF $Z_2 \neq 0$ GOTO A
  GOTO E
  $[A]$ $Z_2 \leftarrow Z_2 - 1$
  $Z_1 \leftarrow X_1 + Y$
  $Y \leftarrow Z_1$
  GOTO B
- OK!

A Shorter Program That Computes $f(x_1, x_2) = x_1 \cdot x_2$?

- $Z_2 \leftarrow X_2$
  $[B]$ IF $Z_2 \neq 0$ GOTO A
  GOTO E
  $[A]$ $Z_2 \leftarrow Z_2 - 1$
  $Y \leftarrow X_1 + Y$
  GOTO B
- *NO GOOD!*
- Why?
- The macro for $f(x_1, x_2) = x_1 + x_2$

  $Y \leftarrow X_1$
  $Z \leftarrow X_2$
  $[B]$ IF $Z \neq 0$ GOTO A
  GOTO E
  $[A]$ $Z \leftarrow Z - 1$
  $Y \leftarrow Y + 1$
  GOTO B

- Macro expanding $Y \leftarrow X_1 + Y$:

$$
\begin{array}{ll}
 & Y \leftarrow X_1 \\
 & Z \leftarrow Y \\
[B] & \text{IF } Z \neq 0 \text{ GOTO } A \\
 & \text{GOTO } E \\
[A] & Z \leftarrow Z - 1 \\
 & Y \leftarrow Y + 1 \\
 & \text{GOTO } B
\end{array}
$$

- The above actually computes $f(x_1, x_2) = 2 \cdot x_1$

A Program That Computes $f(x_1, x_2) = x_1 \cdot x_2$, Revisited

- Need to macro expand $Z_1 \leftarrow X_1 + Y$.
- After macro expansion:

$$
\begin{array}{ll}
 & Z_2 \leftarrow X_2 \\
[B] & \text{IF } Z_2 \neq 0 \text{ GOTO } A \\
 & \text{GOTO } E \\
[A] & Z_2 \leftarrow Z_2 - 1 \\
 & Z_1 \leftarrow X_1 \\
 & Z_3 \leftarrow Y \\
[B_2] & \text{IF } Z_3 \neq 0 \text{ GOTO } A_2 \\
 & \text{GOTO } E_2 \\
[A_2] & Z_3 \leftarrow Z_3 - 1 \\
 & Z_1 \leftarrow Z_1 + 1 \\
 & \text{GOTO } B_2 \\
[E_2] & Y \leftarrow Z_1 \\
 & \text{GOTO } B
\end{array}
$$

Note on The Macro Expansion

- The output variable Y in the macro $f(x_1, x_2) = x_1 + x_2$ is now fresh variable Z_1 in the expanded form.
- The local variable Z in the macro $f(x_1, x_2) = x_1 + x_2$ is now fresh variable Z_3 in the expanded form (as variables Z_1 and Z_2 are already used).
- Fresh labels A_2, B_2, and E_2 are used in the expanded form (as the original labels A, B, and E are already used).

- The instruction GOTO E_2 only terminates the addition. The computation must continue to place following the addition. Hence, the instruction immediately following the addition is labeled E_2.
- *Unlimited supply of fresh local variables and local labels!*
- More about macro expansion next week.

A Final Example

- What does this program compute?

$$
\begin{array}{ll}
 & Y \leftarrow X_1 \\
 & Z \leftarrow X_2 \\
[C] & \text{IF } Z \neq 0 \text{ GOTO } A \\
 & \text{GOTO } E \\
[A] & \text{IF } Y \neq 0 \text{ GOTO } B \\
 & \text{GOTO } A \\
[B] & Y \leftarrow Y - 1 \\
 & Z \leftarrow Z - 1 \\
 & \text{GOTO } C
\end{array}
$$

- If we begin with $X_1 = 5$ and $X_2 = 2$, ...
- If we begin with $X_1 = 2$ and $X_2 = 5$, ...
- This program computes the following *partial function*

$$
g(x_1, x_2) = \begin{cases} x_1 - x_2 & \text{if } x_1 \geq x_2 \\ \uparrow & \text{if } x_1 < x_2 \end{cases}
$$