Denotational Semantics

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This course note . . .

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Course outline

Unit 1. Basic domain theory.

Unit 2. Denotational semantics of functional programs and **While** programs.

Unit 3. Non-standard semantics.

Each unit consists of 2 hours of lecture and 1 hour of lab/tutor.

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- Semantics:
  f is a function computing the square of its argument n;
  k is the result of applying f to integer 10.

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- The denotational semantics and the operational semantics of a programming language shall closely relate to each other.
- *Programs shall have precise and consistent meaning.*

Non-terminating Programs

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Which interpretation is accurate?

Non-terminating Programs, Continued

```
let rec g n = if (n mod 2 = 0)
                then not (g (n+1))
                else not (g (n-1))
```

Which of the following meaning of g is more accurate?

- $g_3(n) = \uparrow$
- $g_4(n) = \begin{cases} \mathrm{T} & \text{if } n = 0, \\ \mathrm{F} & \text{if } n = 1, \\ \uparrow & \text{otherwise.} \end{cases}$
- $g_5(n) = \begin{cases} \mathrm{F} & \text{if } n = 0, \\ \mathrm{T} & \text{if } n = 1, \\ \uparrow & \text{otherwise.} \end{cases}$

Note: We use $\uparrow$ as a shorthand for non-termination or abnormal termination. Functions g_3, g_4, and g_5 are *partial* functions.

A Notation for Functions

For a (partial) function f, we use the notation $f = \{(d, e) \mid f(d) = e, e \text{ is defined}\}$.

$$g_1(n) = \begin{cases} \mathrm{T} & \text{if } n \text{ is even} \\ \mathrm{F} & \text{if } n \text{ is odd} \end{cases} \qquad g_1 = \begin{array}{l} \{(2n, \mathrm{T}) \mid n \geq 0\} \cup \\ \{(2n+1, \mathrm{F}) \mid n \geq 0\} \end{array}$$

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Data Types and Sets

In programming languages, a data type can be viewed as a set of values, along with predefined operations on values in the set.

- For type `int`, we think of the set $Z = \{\ldots, 2, -1, 0, 1, 2, \ldots\}$ along with integer operations $+, -, \times, \div, \ldots$
- For type `bool`, we think of the set $B = \{\mathrm{T}, \mathrm{F}\}$, along with boolean operations $\vee, \wedge, \neg, \ldots$.

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This view, however, does not address non-terminating programs.

- Which element in B gives meaning to `(g 0)`?
- Of the 5 meanings g_1, g_2, g_3, g_4, g_5 for g, which one most accurately describes g?

Data Types and Domains

To address non-termination,

- For each data type, an element $\perp$ is introduced to the set of values to denote computational divergence.
- A partial order is established among the elements in the new set. This set is called the *domain* for the data type.

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- Elements in $\mathcal{B}$ are ordered by $\perp \sqsubseteq \mathrm{T}$ and $\perp \sqsubseteq \mathrm{F}$. But $\mathrm{T} \not\sqsubseteq \mathrm{F}$ and $\mathrm{F} \not\sqsubseteq \mathrm{T}$.

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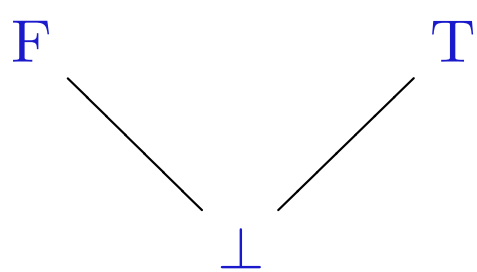
- Domain $\mathcal{B}$ illustrated:



Partially Ordered Set (poset)

Definition

A *partially ordered set* (poset) D is a set with a binary relation $\sqsubseteq_D \subseteq D \times D$ such that for every $x, y, z \in D$, the following properties fold:

1. (reflexive) $x \sqsubseteq_D x$.
2. (anti-symmetric) $x \sqsubseteq_D y$ and $y \sqsubseteq_D x$ implies $x = y$.
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- $B = \{\mathrm{T}, \mathrm{F}\}$ with $\sqsubseteq_B = \{(\mathrm{F}, \mathrm{F}), (\mathrm{T}, \mathrm{T})\}$ is a poset.
- $\mathcal{B} = \{\bot, \mathrm{F}, \mathrm{T}\}$ with $\sqsubseteq_{\mathcal{B}} = \{(\bot, \bot), (\mathrm{F}, \mathrm{F}), (\mathrm{T}, \mathrm{T}), (\bot, \mathrm{F}), (\bot, \mathrm{T})\}$ is also a poset.

Directed Set

Definition

Let D be a poset. A set $X \subseteq D$ is *directed* if

1. $X \neq \emptyset$.
2. For all $x, y \in X$ there is a $z \in X$ such that $x \sqsubseteq_D z$ and $y \sqsubseteq_D z$.

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- A directed set X of a poset D can be viewed as an approximation for some computation in D.
- It is an approximation because for every two elements $x, y \in X$, there is always a more defined element $z \in X$ which x and y can progress to.

Complete Partial Order (cpo)

Definition

Let D be a poset. D is a *complete partial order* (cpo) if

1. There is a least element $\bot_D \in D$ such that for all $x \in D$, $\bot_D \sqsubseteq_D x$.
2. Every directed set $X \subseteq D$ has a *least upper bound* (lub) $\bigsqcup X \in D$.

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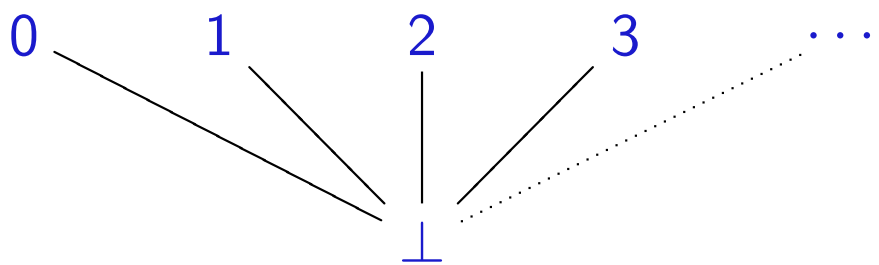
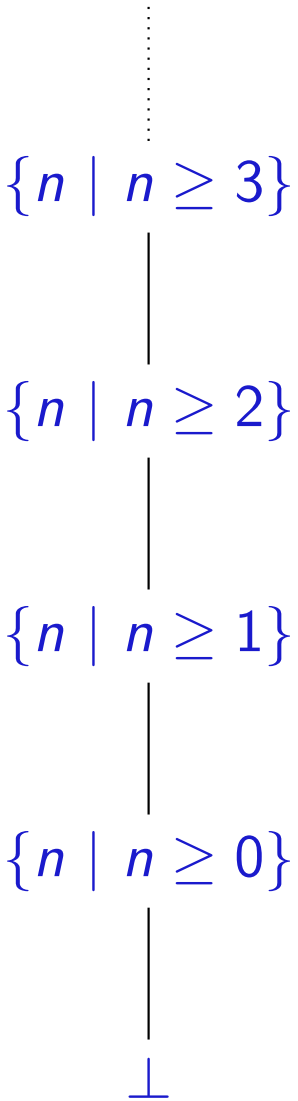
- We use cpo as the domain for denotational semantics. The terms cpo and domain are used interchangeably.
- The subscript D in $\sqsubseteq_D$ and $\bot_D$ is often omitted if it is clear.

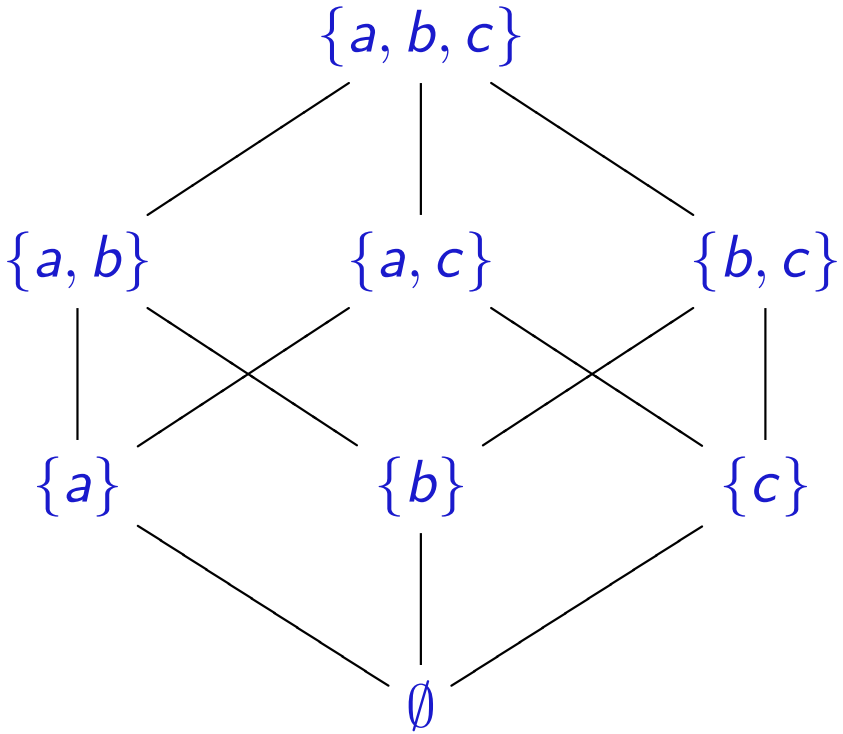
Domain $\mathcal{N}$



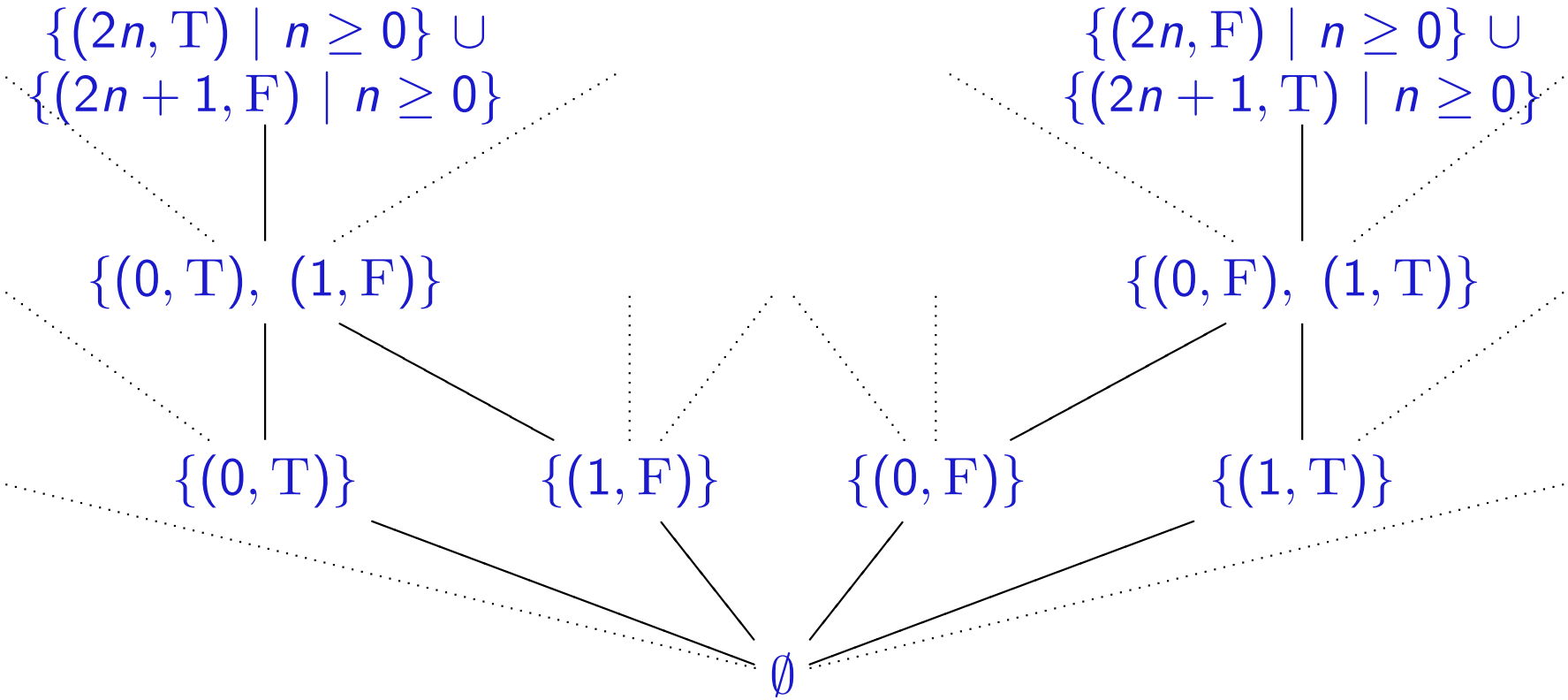
A Possible View of Natural Numbers



A Domain Built from Subsets



A Domain of Partial Functions



Domain Lifting

Definition

Let D be a domain. Define poset $\mathrm{lift}(D)$ by

1. $\mathrm{lift}(D) = D \cup \{\bot_{\mathrm{lift}(D)}\}$, $\bot_{\mathrm{lift}(D)} \notin D$.
2. $x \sqsubseteq_{\mathrm{lift}(D)} y$ if and only if $x = \bot_{\mathrm{lift}(D)}$ or $x \sqsubseteq_D y$.

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□

- If D is a domain then $\mathrm{lift}(D)$ forms a domain.
- $\mathrm{lift}(D)$ is called the *lifted* domain of D.

Domain *1* and Domain *2*

Example

Let *1* be the poset $\{\bot\}$ where $\bot \sqsubseteq_1 \bot$. *1* is a domain. □

Domain *1* and Domain *2*

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Example

Let $2 = \mathrm{lift}(1)$. Then *2* is a domain. *2* has only two elements, $\bot$ and $\top$, and $\bot \sqsubseteq_2 \top$. □

Domain *1* and Domain *2*

Example

Let 1 be the poset $\{\bot\}$ where $\bot \sqsubseteq_1 \bot$. 1 is a domain. □

Example

Let $2 = \mathrm{lift}(1)$. Then 2 is a domain. 2 has only two elements, $\bot$ and $\top$, and $\bot \sqsubseteq_2 \top$. □

For domain 2, the least upper bound operator $\sqcup$ and the greatest lower bound operator $\sqcap$ are defined by the following.

$\sqcup$	$\bot$	$\top$
$\bot$	$\bot$	$\top$
$\top$	$\top$	$\top$

$\sqcap$	$\bot$	$\top$
$\bot$	$\bot$	$\bot$
$\top$	$\bot$	$\top$

(Look familiar?)

The Sum of Two Domains

Definition

Let D and D' be domains. Define poset $D + D'$ by

1. $D + D' = D \cup D' \cup \{\bot_{D+D'}\}$, where the elements in D are made distinct from the elements in D'. $\bot_{D+D'} \notin D \cup D'$.
2. $x \sqsubseteq_{D+D'} y$ if and only if $x = \bot_{D+D'}$ or $x \sqsubseteq_D y$ or $x \sqsubseteq_{D'} y$.

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□

- If both D and D' are domains, then $D + D'$ is a domain too.
- $D + D'$ is called the *sum* of D and D'.

The Coalesced Sum of Two Domains

Definition

Let D and D' be domains. Define poset $D \oplus D'$ by

1. $D \oplus D' = D \cup D' \cup \{\bot_{D \oplus D'}\}$, where the elements in D are made distinct from the elements in D', except $\bot_D$ and $\bot_{D'}$. $\bot_{D \oplus D'} = \bot_D = \bot_{D'}$.
2. $x \sqsubseteq_{D \oplus D'} y$ if and only if $x = \bot_{D \oplus D'}$ or $x \sqsubseteq_D y$ or $x \sqsubseteq_{D'} y$.

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2. $x \sqsubseteq_{D \oplus D'} y$ if and only if $x = \bot_{D \oplus D'}$ or $x \sqsubseteq_D y$ or $x \sqsubseteq_{D'} y$.

□

- If both D and D' are domains, then $D \oplus D'$ is a domain too.
- $D \oplus D'$ is called the *coalesced sum* of D and D'.

The Product of Two Domains

Definition

Let D and D' be domains. Define $D \times D'$ by

1. $D \times D' = \{ \langle d, d' \rangle \mid d \in D,\ d' \in D' \}$,
   $\bot_{D \times D'} = \langle \bot_D, \bot_{D'} \rangle$.
2. $\langle d_1, d_1' \rangle \sqsubseteq_{D \times D'} \langle d_2, d_2' \rangle$ if and only if $d_1 \sqsubseteq_D d_2$ and $d_1' \sqsubseteq_{D'} d_2'$.

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□

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The Smash Product of Two Domains

Definition

Let D and D' be domains. Define $D \otimes D'$ by

1. $D \otimes D' = \{ \langle d, d' \rangle \mid d \in D,\ d' \in D' \}$.
   $\bot_{D \otimes D'} = \langle \bot_D, d' \rangle = \langle d, \bot_{D'} \rangle$ for any $d \in D$ and $d' \in D'$.
2. $\langle d_1, d_1' \rangle \sqsubseteq_{D \otimes D'} \langle d_2, d_2' \rangle$ if and only if $\langle d_1, d_1' \rangle = \bot_{D \otimes D'}$, or $d_1 \sqsubseteq_D d_2$ and $d_1' \sqsubseteq_{D'} d_2'$.

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Continuous Function

Definition

Let D and D' be domains, and f be a *total* function from D to D'.

1. f is *monotonic* if and only if $f(d_1) \sqsubseteq_{D'} f(d_2)$ whenever $d_1 \sqsubseteq_D d_2$.
2. f is *continuous* if and only if $f(\bigsqcup X) = \bigsqcup f\{X\}$ for every directed set $X \subseteq D$, where $f\{X\}$ is defined as $\{f(x) \mid x \in X\}$.
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□

- If a function f is not strict, it is called *non–strict*.
- If a function is continuous then it is monotonic, but the reverse is not true.

Continuous Function Space

Definition

Let D and D' be domains. Define $D \to D'$ by

1. $D \to D' = \{f \mid f \text{ is a continuous function from } D \text{ to } D'\}$, and $\bot_{D \to D'} = \{(d, \bot_{D'}) \mid d \in D\}$.
2. $f \sqsubseteq_{D \to D'} g$ if and only if for all $d \in D, f(d) \sqsubseteq_{D'} g(d)$.

□

Continuous Function Space as Domain, I

Theorem (Scott)

The continuous function space $D \to D'$ is a domain if both D and D' are domains. □

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$$F^{\sqcup} = \{(d, \bigsqcup \{f(d) \mid f \in F\}) \mid d \in D\}$$

Since F is directed, we know that, for any $d \in D$, $\{f(d) \mid f \in F\} \subseteq D'$ is directed as well. Because D' is a domain, $\bigsqcup \{f(d) \mid f \in F\}$ exists hence function $F^{\sqcup}$ is well defined. Moreover, by construction, we observe that $F^{\sqcup}$ is the lub of F.
(To be continued) □

Continuous Function Space as Domain, II

Proof (Continued).

Why is $\{f(d) \mid f \in F\} \subseteq D'$ directed, and why is $F^{\sqcup}$ the lub of F?

Continuous Function Space as Domain, II

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Why is $\{f(d) \mid f \in F\} \subseteq D'$ directed, and why is $F^{\sqcup}$ the lub of F?

Let $u, v \in \{f(d) \mid f \in F\}$. We have $u = f\ d$ and $v = g\ d$ for some $f, g \in F$. As F is directed, there is a $h \in F$ such that $f \sqsubseteq_{D \to D'} h$ and $g \sqsubseteq_{D \to D'} h$. That is, $f\ d \sqsubseteq_{D'} h\ d$ and $g\ d \sqsubseteq_{D'} h\ d$. Since $h\ d \in \{f(d) \mid f \in F\}$, we conculde the set is directed.

Continuous Function Space as Domain, II

Proof (Continued).

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We first observe that $f \sqsubseteq_{D \to D'} F^{\sqcup}$ for all $f \in F$. That is, $F^{\sqcup}$ is an upper bound of F. Suppose w is also an upper bound of F. That is, $f \sqsubseteq_{D \to D'} w$ for all $f \in F$; hence, for any $d \in D$, $f\ d \sqsubseteq_{D'} w\ d$. Taking the lub at both sides of $\sqsubseteq$, we arrive at $\bigsqcup\{f(d) \mid f \in F\} = F^{\sqcup}\ d \sqsubseteq_{D'} w\ d$ for any $d \in D$. That is, $F^{\sqcup}$ is the lub of F. $\square$

Continuous Function Space as Domain, III

Proof (Continued).

Is $F^{\sqcup}$ a continuous function? For all directed set $X \subseteq D$, we have

$$
\begin{aligned}
& F^{\sqcup}(\bigsqcup X) & \\
= \quad & \bigsqcup_{f \in F} f(\bigsqcup X) & \text{(Definition of } F^{\sqcup}\text{)} \\
= \quad & \bigsqcup_{f \in F}(\bigsqcup_{x \in X} f(x)) & (X \subseteq D \text{ is directed; each } f \text{ is continuous}) \\
= \quad & \bigsqcup_{x \in X}(\bigsqcup_{f \in F} f(x)) & \text{(Rearranging indices)} \\
= \quad & \bigsqcup_{x \in X} F^{\sqcup}(x) & \text{(Definition of } F^{\sqcup}\text{)} \\
= \quad & \bigsqcup F^{\sqcup}\{X\} & \text{(Definition of } \bigsqcup X\text{)}
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We conclude $F^{\sqcup}$ is continuous. □

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We conclude $F^{\sqcup}$ is continuous. □

From now on, we write $\bigsqcup F$ to denote the function $F^{\sqcup}$, the least upper bound of a directed set of continuous functions F.

Why Continuous Function?

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- In particular, we don't want to admit (non-continuous) functions that "jump" arbitrarily at the limit of an approximation.
- Continuous function spaces are themselves complete partial orders so work well with other semantic domains.

Fixed Points and The Least Fixed Point

Definition

Let D be a poset and let $f \in D \to D$ be a total function.

1. $x \in D$ is a *fixed point* of f if and only if $f(x) = x$.
2. x is the *least fixed point* of f if and only if x is a fixed point of f, and for every fixed point $d \in D$ of f, it implies $x \sqsubseteq_D d$.

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□

- Function $f(x) = x$, where $x \in \mathcal{B}$, have three fixed points: $\bot$, F, and T.
- $\bot$ is the least fixed point of f.

The Least Fixed Point Theorem

Theorem (Kleene)

Let D be a domain.

1. *Every function $f \in D \to D$ has a least fixed point.*
2. *There exists a function $\mathrm{fix} \in (D \to D) \to D$ such that for every function $f \in D \to D$, $\mathrm{fix}\,(f)$ is the least fixed point of f.*

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Proof.

1. $X_f = \{\bot_D,\ \ f(\bot_D),\ \ f(f(\bot_D)),\ \ \ldots,\ \ f^{(n)}(\bot_D),\ \ \ldots\}$ is a directed set because $\bot_D \sqsubseteq_D f(\bot_D)$, $f(\bot_D) \sqsubseteq_D f(f(\bot_D))$, $\ldots$.
By the continuity of f,

$$f(\bigsqcup X_f) = \bigsqcup f\{X_f\} = \bigsqcup X_f$$

Hence, $\bigsqcup X_f$ is a fixed point of f. (To be continued) □

The Least Fixed Point Theorem, Continued

Proof (Continued).

Moreover, suppose that d too is a fixed point of f. Then

$$\bot_D \sqsubseteq_D d,\ f(\bot_D) \sqsubseteq_D f(d) = d,\ \ldots,\ f^{(n)}(\bot_D) \sqsubseteq_D f^{(n)}(d) = d,\ \ldots$$

Taking the lub of both sides, it follows that $\bigsqcup X_f \sqsubseteq_D d$.

The Least Fixed Point Theorem, Continued

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Taking the lub of both sides, it follows that $\bigsqcup X_f \sqsubseteq_D d$.

2. Define function *fix* by

$$fix\,(f) = \bigsqcup X_f$$

Then $fix\,(f)$ is the least fixed point of f. Moreover, by rearranging indices, we can show that, for all directed set $F \subseteq D \to D$

$$fix\,(\bigsqcup F) = \bigsqcup\ fix\{F\}$$

That is, f is continuous hence $f \in (D \to D) \to D$. □

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- We then assign $g = fix(f) \in \mathcal{N} \to \mathcal{B}$ as the meaning of g.

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  - `()` is mapped to $\top \in 2$, `true` is mapped to $\mathrm{T} \in \mathcal{B}$, `not` is mapped to a function $\mathit{not} \in \mathcal{B} \to \mathcal{B}$ where $\mathit{not} = \{(\bot, \bot),\ (\mathrm{F}, \mathrm{T}),\ (\mathrm{T}, \mathrm{F})\}$.

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  - We write $[\![()]\!]_2 = \top$, $[\![\texttt{true}]\!]_{\mathcal{B}} = \mathrm{T}$, and $[\![\texttt{not}]\!]_{\mathcal{B}\to\mathcal{B}} = \mathit{not}$, etc.

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- For a user-defined term, construct a semantic equation based on its definition, reusing existing terms and constants.
- If the definition is recursive, compute the least fixed point.

Compose Meaning for Programs, An Example

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We compose the following function $f \in (\mathcal{N} \to \mathcal{B}) \to (\mathcal{N} \to \mathcal{B})$:

$$f\ g\ n = \mathit{if\text{-}then\text{-}else}\ (\mathit{eq}\ (\mathit{mod}\ n\ 2)\ 0)\ (\mathit{not}\ (g\ (\mathit{plus}\ n\ 1)))\ (\mathit{not}\ (g\ (\mathit{minus}\ n\ 1)))$$

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where functions

$$\begin{aligned} \mathit{if\text{-}then\text{-}else} &\in \mathcal{B} \to \mathcal{N} \to \mathcal{N} \to \mathcal{N} \\ \mathit{eq} &\in \mathcal{N} \to \mathcal{N} \to \mathcal{B} \\ \mathit{not} &\in \mathcal{B} \to \mathcal{B} \\ \mathit{mod}, \mathit{plus}, \mathit{minus} &\in \mathcal{N} \to \mathcal{N} \to \mathcal{N} \end{aligned}$$

if-then-else	$\bot$	x	y	$=$	$\bot$
if-then-else	T	x	y	$=$	x
if-then-else	F	x	y	$=$	y
eq	$\bot$	y		$=$	$\bot$
eq	x	$\bot$		$=$	$\bot$
eq	x	y		$=$	T
eq	x	y		$=$	F
not	$\bot$			$=$	$\bot$
not	F			$=$	T
not	T			$=$	F
$\ldots$					
minus	$\bot$	y		$=$	$\bot$
minus	x	$\bot$		$=$	$\bot$
minus	x	y		$=$	$\bot$
minus	x	y		$=$	$x - y$

where $x = y \neq \bot$

otherwise

where $x < y$

otherwise

The Least Fixed Point Iteration

Start with $\bot_{\mathcal{N}\to\mathcal{B}} = \{(n, \bot) \mid n \in \mathcal{N}\}$, we compute the least upper bound of the following directed set

$$\{\bot_{\mathcal{N}\to\mathcal{B}}, \quad f(\bot_{\mathcal{N}\to\mathcal{B}}), \quad f(f(\bot_{\mathcal{N}\to\mathcal{B}})), \quad \ldots\}$$

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Note that $f(\bot_{\mathcal{N}\to\mathcal{B}})$ computes to

$$g\ n\ =\ \mathit{if\text{-}then\text{-}else}\ (\mathit{eq}\ (\mathit{mod}\ n\ 2)\ 0)\ \bot\ \bot$$

This is simplified to

$$g\ n\ =\ \bot$$

The Least Fixed Point Iteration

Start with $\bot_{\mathcal{N}\to\mathcal{B}} = \{(n, \bot) \mid n \in \mathcal{N}\}$, we compute the least upper bound of the following directed set

$$\{\bot_{\mathcal{N}\to\mathcal{B}},\quad f(\bot_{\mathcal{N}\to\mathcal{B}}),\quad f(f(\bot_{\mathcal{N}\to\mathcal{B}})),\quad \ldots\}$$

Note that $f(\bot_{\mathcal{N}\to\mathcal{B}})$ computes to

$$g\ n\ =\ \mathit{if\text{-}then\text{-}else}\ (\mathit{eq}\ (\mathit{mod}\ n\ 2)\ 0)\ \bot\ \bot$$

This is simplified to

$$g\ n\ =\ \bot$$

That is, we have reached $\bot_{\mathcal{N}\to\mathcal{B}}$ as the least fixed point of f. We conclude that

$$[\![\mathrm{g}]\!]_{\mathcal{N}\to\mathcal{B}} = \bot_{\mathcal{N}\to\mathcal{B}} = \{(n, \bot) \mid n \in \mathcal{N}\}$$

That is, g will not terminate for any given input.

The Factorial Program

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Start with $\bot_{\mathcal{N}\to\mathcal{N}} = \{(n, \bot) \mid n \in \mathcal{N}\}$, the least fixed point iteration for f will be

$$\begin{aligned}
f^{(0)}(\bot_{\mathcal{N}\to\mathcal{B}}) &= \{(n, \bot) \mid n \in \mathcal{N}\} \\
f^{(1)}(\bot_{\mathcal{N}\to\mathcal{B}}) &= \{(0,1)\} \cup \{(n, \bot) \mid n \notin \{0\}\} \\
f^{(2)}(\bot_{\mathcal{N}\to\mathcal{B}}) &= \{(0,1),\ (1,1)\} \cup \{(n, \bot) \mid n \notin \{0,1\}\} \\
f^{(3)}(\bot_{\mathcal{N}\to\mathcal{B}}) &= \{(0,1),\ (1,1),\ (2,2)\} \cup \{(n, \bot) \mid n \notin \{0,1,2\}\} \\
&\vdots \\
f^{(k+1)}(\bot_{\mathcal{N}\to\mathcal{B}}) &= \{(n, n!) \mid n \leq k\} \cup \{(n, \bot) \mid n \notin \{0,1,\ldots,k\}\} \\
&\vdots
\end{aligned}$$

Dealing With Mutual Recursion

For functions even and odd defined as

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let rec even n = if n=0 then true  else odd  (n-1)
    and odd  n = if n=0 then false else even (n-1)
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We first define the following (non-recursive) function $f \in (\mathcal{N} \to \mathcal{B}) \times (\mathcal{N} \to \mathcal{B}) \to (\mathcal{N} \to \mathcal{B}) \times (\mathcal{N} \to \mathcal{B})$:

$$f\ (even,\ odd) =$$
$$(\{(n, \mathit{if\text{-}then\text{-}else}\ (\mathit{eq}\ n\ 0)\ \mathrm{T}\ (\mathit{odd}\ (\mathit{minus}\ n\ 1))) \mid n \in \mathcal{N}\},$$
$$\{(n, \mathit{if\text{-}then\text{-}else}\ (\mathit{eq}\ n\ 0)\ \mathrm{F}\ (\mathit{even}\ (\mathit{minus}\ n\ 1))) \mid n \in \mathcal{N}\})$$

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Note that the least fixed point of f is a pair of functions $(even, odd)$ mutually satisfying

$$\begin{aligned} even &= \{(n, \mathit{if\text{-}then\text{-}else}\ (\mathit{eq}\ n\ 0)\ \mathrm{T}\ (\mathit{odd}\ (\mathit{minus}\ n\ 1))) \mid n \in \mathcal{N}\} \\ odd &= \{(n, \mathit{if\text{-}then\text{-}else}\ (\mathit{eq}\ n\ 0)\ \mathrm{F}\ (\mathit{even}\ (\mathit{minus}\ n\ 1))) \mid n \in \mathcal{N}\} \end{aligned}$$

Dealing With Mutual Recursion, Continued

We then start the least fixed point iteration with $(\bot_{\mathcal{N}\to\mathcal{B}},\ \bot_{\mathcal{N}\to\mathcal{B}})$, and get

	$\bot$	0	1	2	3	...
$even^{(0)}$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	...
$odd^{(0)}$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	...
$even^{(1)}$	$\bot$	T	$\bot$	$\bot$	$\bot$	...
$odd^{(1)}$	$\bot$	F	$\bot$	$\bot$	$\bot$	...
$even^{(2)}$	$\bot$	T	F	$\bot$	$\bot$	...
$odd^{(2)}$	$\bot$	F	T	$\bot$	$\bot$	...
$even^{(3)}$	$\bot$	T	F	T	$\bot$	...
$odd^{(3)}$	$\bot$	F	T	F	$\bot$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Modeling States of **While** Programs

- The execution of a **While** program results in a change to the state of the machine. A state is a mapping from variables to values where undefined variables are mapped to $\perp$.
- A state can be queried and updated. Let s be a state, x be a variable, and v be a value. Then,
  - $s\ x$ returns the value associated to variable x in state s.
  - $s[x \mapsto v]$ is the state identical to s except now variable x is mapped to v.
- Let $s = [x \mapsto 5,\ y \mapsto 7,\ z \mapsto 0]$, then
  - $s\ y$ is 7,
  - $s[x \mapsto 3]$ is $[x \mapsto 3,\ y \mapsto 7,\ z \mapsto 0]$.

Evaluation at the Presence of States

The semantics of arithmetic and boolean expressions in **While** programs is now defined by evaluation at the presence of states. As an example, let state $s = [\mathrm{x} \mapsto 3,\ \mathrm{y} \mapsto 7,\ \mathrm{z} \mapsto 0]$, then

$$\begin{aligned} [\![\mathrm{x}\ +\ 1]\!]_{\mathcal{N}}\ s &= [\![\mathrm{x}]\!]_{\mathcal{N}}\ s + [\![1]\!]_{\mathcal{N}}\ s \\ &= (s\ \mathrm{x}) + [\![1]\!]_{\mathcal{N}} \\ &= 3 + 1 \\ &= 4 \end{aligned}$$

and

$$\begin{aligned} [\![\neg(\mathrm{x}\ =\ 1)]\!]_{\mathcal{B}}\ s &= \mathit{not}\ ([\![\mathrm{x}\ =\ 1]\!]_{\mathcal{B}}\ s) \\ &= \mathit{not}\ ([\![\mathrm{x}]\!]_{\mathcal{N}}\ s\ =\ [\![1]\!]_{\mathcal{N}}\ s) \\ &= \mathit{not}\ ((s\ \mathrm{x}) = [\![1]\!]_{\mathcal{N}}) \\ &= \mathit{not}\ (3 = 1)\ =\ \mathit{not}\ \mathrm{F}\ =\ \mathrm{T} \end{aligned}$$

Semantics of **While** statements

- $[\![x := a]\!]_{\mathcal{S}}\ s = s[x \mapsto [\![a]\!]_{\mathcal{N}}\ s]$
- $[\![\texttt{skip}]\!]_{\mathcal{S}} = \mathrm{id}$
- $[\![S_1 ; S_2]\!]_{\mathcal{S}} = [\![S_2]\!]_{\mathcal{S}} \circ [\![S_1]\!]_{\mathcal{S}}$
- $[\![\texttt{if } b \texttt{ then } S_1 \texttt{ else } S_2]\!]_{\mathcal{S}} = \mathrm{cond}\ ([\![b]\!]_{\mathcal{B}},\ [\![S_1]\!]_{\mathcal{S}},\ [\![S_2]\!]_{\mathcal{S}})$
- $[\![\texttt{while } b \texttt{ do } S]\!]_{\mathcal{S}} = \mathrm{fix}\ F$
  where $F\ g = \mathrm{cond}\ ([\![b]\!]_{\mathcal{B}},\ g \circ [\![S]\!]_{\mathcal{S}},\ \mathrm{id})$

where formally

- $\mathcal{S}$ is the domain **State** $\to$ **State**,
- s is an element in domain **State**, and
- **State** is the mapping from variables to values.

Note that our notations are different from those in the textbook. Also $\mathcal{S}$ (and **State**, when viewed as a function) are continuous instead of being partial; $\perp_{\textbf{State}} = \{(x, \perp) \mid x \text{ is a variable}\}$.

Denotational Semantics of Assignment and Skip

$[\![x := e]\!]_{\mathcal{S}}\ s\ =\ s[x \mapsto [\![e]\!]_{\mathcal{N}}\ s]$

- compute the semantics of expression e at state s, which is an element in domain $\mathcal{N}$;
- map variable x to this element; and
- update the state s with the above mapping.

$[\![\texttt{skip}]\!]_{\mathcal{S}}\ =\ \mathrm{id}$

- id is the identity function: $\mathrm{id}\ x\ =\ x$;
- `skip` has no effect on the state: for all state s, $[\![\texttt{skip}]\!]_{\mathcal{S}}\ s\ =\ s$;
- which means $[\![\texttt{skip}]\!]_{\mathcal{S}}$ is the identity function!

Denotational Semantics of Sequencer and Conditional

$[\![S_1 \ ; \ S_2]\!]_{\mathcal{S}} \ = \ [\![S_2]\!]_{\mathcal{S}} \circ [\![S_1]\!]_{\mathcal{S}}$

- for all state s, $([\![S_2]\!]_{\mathcal{S}} \circ [\![S_1]\!]_{\mathcal{S}})\ s = [\![S_2]\!]_{\mathcal{S}}\ ([\![S_1]\!]_{\mathcal{S}}\ s)$;
- the result is s'' whenever $[\![S_1]\!]_{\mathcal{S}}\ s = s'$ and $[\![S_2]\!]_{\mathcal{S}}\ s' = s''$;
- note that if either s' or s'' is $\perp_{\mathcal{S}}$, the end result is also $\perp_{\mathcal{S}}$.

$[\![\texttt{if}\ b\ \texttt{then}\ S_1\ \texttt{else}\ S_2]\!]_{\mathcal{S}} \ = \ \texttt{cond}\ ([\![b]\!]_{\mathcal{B}},\ [\![S_1]\!]_{\mathcal{S}},\ [\![S_2]\!]_{\mathcal{S}})$

- $\texttt{cond}$ is a function in domain $(\textbf{State} \to \mathcal{B}) \times \mathcal{S} \times \mathcal{S} \to \mathcal{S}$;
- it is defined by

$$\texttt{cond}\ (p, g_1, g_2)\ s = \begin{cases} g_1\ s & \text{if } p\ s = \mathrm{T} \\ g_2\ s & \text{if } p\ s = \mathrm{F} \end{cases}$$

  where s is a state;
- note that the result is $\perp_{\textbf{State}}$ if $(p\ s = \perp_{\mathcal{B}})$ or $(p\ s = \mathrm{T}$ and $g_1\ s = \perp_{\textbf{State}})$ or $(p\ s = \mathrm{F}$ and $g_2\ s = \perp_{\textbf{State}})$.

Denotational Semantics of While Loop

$[\![\texttt{while}\ b\ \texttt{do}\ S]\!]_{\mathcal{S}} = \text{fix}\ F$
where $F\ g = \texttt{cond}\ ([\![b]\!]_{\mathcal{B}},\ g \circ [\![S]\!]_{\mathcal{S}},\ \text{id})$

- observe that $\texttt{while}\ b\ \texttt{do}\ S$ has the same effect of $\texttt{if}\ b\ \texttt{then}\ (S\ ;\ \texttt{while}\ b\ \texttt{do}\ S)\ \texttt{else}\ \texttt{skip}$
- the two must have the same semantics:
  $[\![\texttt{while}\ b\ \texttt{do}\ S]\!]_{\mathcal{S}} = \texttt{cond}\ ([\![b]\!]_{\mathcal{B}},\ [\![\texttt{while}\ b\ \texttt{do}\ S]\!]_{\mathcal{S}} \circ [\![S]\!]_{\mathcal{S}},\ \text{id})$
- $[\![\texttt{while}\ b\ \texttt{do}\ S]\!]_{\mathcal{S}}$ is a fixed point of the functional F:
  $F\ g = \texttt{cond}\ ([\![b]\!]_{\mathcal{B}},\ g \circ [\![S]\!]_{\mathcal{S}},\ \text{id})$
- note that F is an element in domain $\mathcal{S} \to \mathcal{S}$, and
- the fixed point function fix is an element in domain $(\mathcal{S} \to \mathcal{S}) \to \mathcal{S}$.

Which Fixed Point? An Exmaple, I

Consider this program: $\texttt{while}\ \neg(\texttt{x} = 0)\ \texttt{do}\ \texttt{skip}$

- the corresponding functional F is defined by

$$F\ g\ s = \texttt{cond}\ (\llbracket\neg(\texttt{x} = 0)\rrbracket_{\mathcal{B}},\ g \circ \llbracket\texttt{skip}\rrbracket_{\mathcal{S}},\ \text{id})\ s$$
$$= \begin{cases} g\ s & \text{if } s\ \texttt{x} \neq 0 \\ s & \text{if } s\ \texttt{x} = 0 \end{cases}$$

- function $g_0\ s = \begin{cases} \perp_{\mathbf{State}} & \text{if } s\ \texttt{x} \neq 0 \\ s & \text{if } s\ \texttt{x} = 0 \end{cases}$ is a fixed point of F,
- so is $g_1\ s = \begin{cases} \perp_{\mathbf{State}} & \text{if } s\ \texttt{x} \notin \{0,1\} \\ s & \text{if } s\ \texttt{x} \in \{0,1\} \end{cases}$ a fixed point, and
- so is $g_2\ s = \begin{cases} \perp_{\mathbf{State}} & \text{if } s\ \texttt{x} \notin \{0,1,2\} \\ s & \text{if } s\ \texttt{x} \in \{0,1,2\} \end{cases}$, and so on.

Clearly we want g_0 as the semantics of this program. How?

Which Fixed Point? An Exmaple, II

What is the least fixed point of functional F:

$$F\ g\ s \quad = \quad \begin{cases} g\ s & \text{if } s\ \mathtt{x} \neq 0 \\ s & \text{if } s\ \mathtt{x} = 0 \end{cases}$$

where $F \in \mathcal{S} \to \mathcal{S}$, $g \in \mathcal{S} = \mathbf{State} \to \mathbf{State}$, and $s \in \mathbf{State}$.
Start with $\bot_{\mathcal{S}}$ which is defined as

$$\bot_{\mathcal{S}}\ x \quad = \quad \bot_{\mathbf{State}}, \quad \text{for all } x \in \mathbf{State}$$

the least fixed point iteration for F will be

$$F^{(0)}(\bot_{\mathcal{S}})\ s \quad = \quad \bot_{\mathcal{S}}\ s = \bot_{\mathbf{State}}$$

$$F^{(1)}(\bot_{\mathcal{S}})\ s \quad = \quad \begin{cases} \bot_{\mathbf{State}} & \text{if } s\ \mathtt{x} \neq 0 \\ s & \text{if } s\ \mathtt{x} = 0 \end{cases}$$

$$F^{(2)}(\bot_{\mathcal{S}})\ s \quad = \quad \begin{cases} \bot_{\mathbf{State}} & \text{if } s\ \mathtt{x} \neq 0 \\ s & \text{if } s\ \mathtt{x} = 0 \end{cases}$$

Clearly the least fixed point is reached at $F^{(2)}(\bot_{\mathcal{S}})$.

Does the Least Fixed Point Always Exist?

Do all **While** programs have well–defined denotational semantics?
We just need to ensure that all semantic functions are continuous!
In particular,

- the semantic functions for all primitive arithmetic and boolean operators are continuous,
- the conditional function is continuous,
- that function composition is continuous, and
- the fixed point function is continuous!

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  - Will `(f x)` always terminate for all `x`?
- For built-in data types and constants, we may use non-standard domains and their elements. For example, we may interpret `if ... then ... else ...` as

$$\begin{array}{llllll} \textit{if-then-else} & \{\bot\} & X & Y & = & \{\bot\} \\ \textit{if-then-else} & B & X & Y & = & X \cup Y \quad \text{otherwise} \end{array}$$

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- However, we need to be precise about these non-standard domains too!

Scott-closed Set

Definition

Let D be a domain. A set $X \subseteq D$ is *Scott–closed* if

1. If $Y \subseteq X$ and Y is directed, then $\bigsqcup Y \in X$.
2. If $x \in X, y \sqsubseteq_D x$, then $y \in X$.

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The least Scott–closed set containing a set Y is written as Y^*.

Hoare Power Domain

Definition

Let D be a domain. Define $\mathrm{P}(D)$ by

1. $\mathrm{P}(D) = \{S \mid \emptyset \neq S \subseteq D,\ S \textit{ is Scott–closed }\}$, and $\bot_{\mathrm{P}(D)} = \{\bot_D\}$.
2. $S \sqsubseteq_{\mathrm{P}(D)} T$ if and only if $S \subseteq T$.

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- If D is a domain, then $\mathrm{P}(D)$ is a domain too. It is called the *Hoare power domain*.
- Not only can we apply P to domains, we can apply it to continuous functions as well. For a function $f \in D \to E$, the function $\mathrm{P}(f) \in \mathrm{P}(D) \to \mathrm{P}(E)$ is defined as

$$\mathrm{P}(f)\ (X) = \{f(x) \mid x \in X\}^*$$

Mappings between A Domain and Its Hoare Power Domain

- The function $\{\cdot\} \in D \to \mathrm{P}(D)$ is defined by

$$\{d\} = \{d\}^*$$

- For a function from $\mathrm{P}(D)$ to D, we can use the least upper bound function $\bigsqcup X$, where $X \in \mathrm{P}(D)$, if D is a complete lattice.

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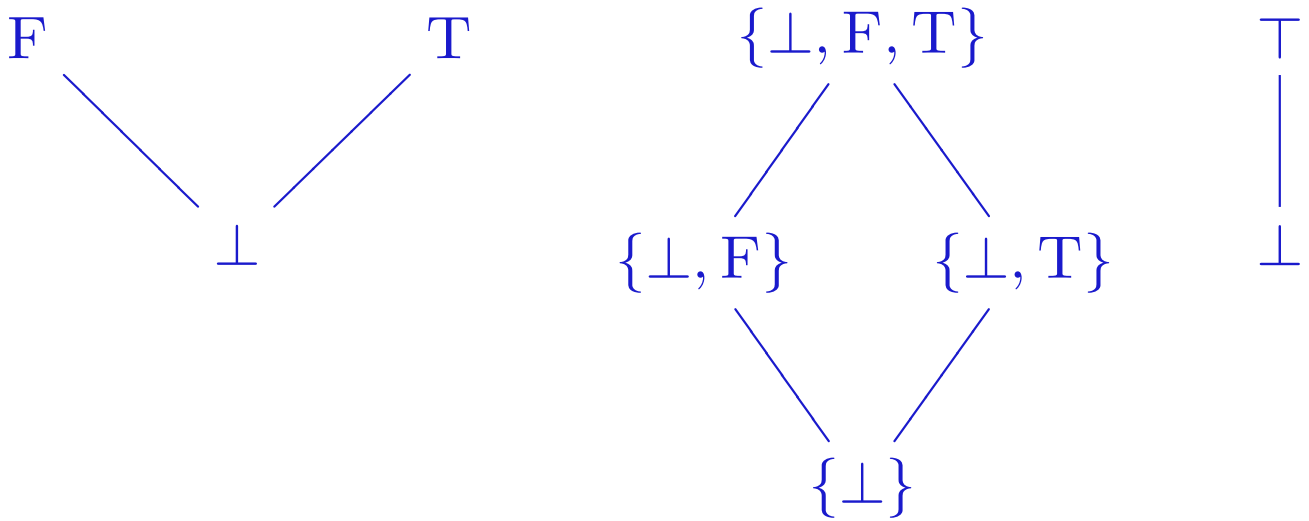
- A complete lattice is a poset in which all subsets have a least upper bound.
- Note that both $\{\cdot\}$ and $\bigsqcup$ are continuous.

Concrete Domain, Hoare Power Domain, and Abstract Domain

Take $\mathcal{B}$ as an example. From $\mathcal{B}$ we can build the Hoare power domain $\mathrm{P}(\mathcal{B})$. We too can reduce $\mathcal{B}$ to a two-element abstract domain $\bar{\mathcal{B}} = 2$. Relative to its Hoare power domain $\mathrm{P}(\mathcal{B})$ and its abstract domain $\bar{\mathcal{B}}$, we call $\mathcal{B}$ the concrete domain, or the standard domain.



Collecting Interpretation and Abstract Interpretation

- Instead of using the standard domains, we can map data types to Hoare power domains, and map programs to functions between the Hoare power domains. When so doing, we are performing collecting interpretation of functional programs.

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- Instead of using the standard domains, we can map data types to abstract domains, and map programs to functions between the abstract domains. When so doing, we are performing abstract interpretation of functional programs.
- Abstract interpretation is a useful technique for program analysis, but we need to relate the three semantics: standard interpretation, collecting interpretation, and abstract interpretation.

Strictness Analysis

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- In O'Caml, the evaluation for `zero` will diverge.

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let rec loop  x = loop x
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- For call-by-name/need languages, strictness analysis is used to determine if functions in a program are strict or not.

Language Constructs May Be Non-strict

For the `if ... then ... else ...` language construct (in both call-by-value and call-by-name/need languages), we define function $\mathit{if\text{-}then\text{-}else} \in \mathcal{B} \to \mathcal{N} \to \mathcal{N} \to \mathcal{N}$ below as its semantics:

$$\begin{array}{llllll} \mathit{if\text{-}then\text{-}else} & \bot & x & y & = & \bot \\ \mathit{if\text{-}then\text{-}else} & \mathrm{T} & x & y & = & x \\ \mathit{if\text{-}then\text{-}else} & \mathrm{F} & x & y & = & y \end{array}$$

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Note that *if-then-else* is strict in its first argument, non-strict in its third argument if its first argument is T, and non-strict in its second argument if its first argument is F.

Abstract Interpretation for Strictness Analysis

- For domains such as $\mathcal{B}$ and $\mathcal{N}$, we now use 2 as the abstract domain with the intention that $\bot$ denotes non-termination while $\top$ denotes values that may or may not terminate.

Abstract Interpretation for Strictness Analysis

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- For domains such as $\mathcal{N} \to \mathcal{N}$, we now use the abstract domain $2 \to 2$ below to denote all possible strictness properties for all elements in $\mathcal{N} \to \mathcal{N}$ (which are continuous functions):

$$\{(\bot, \top),\ (\top, \top)\}$$
$$|$$
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$$|$$
$$\{(\bot, \bot),\ (\top, \bot)\}$$

Abstract Interpretation for Strictness Analysis, Continued

- Constant like *if-then-else* is now a function in the domain $2 \to 2 \to 2 \to 2$. It is defined by

$$\textit{if-then-else}\ b\ x\ y \quad = \quad b \sqcap (x \sqcup y)$$

  Note: An *if-then-else* expression will not terminate if the conditional part b will not terminate, or if both branches x and y will not terminate.
- For a user-defined term, construct an abstract semantic equation based on its definition, and from the abstract semantics of existing terms and constants.
- If the definition is recursive, compute the least fixed point.

Abstract Semantics for Strictness Analysis

$$\begin{aligned} \mathit{if\text{-}then\text{-}else}\ b\ x\ y &= b \sqcap (x \sqcup y) \\ \mathit{eq}\ x\ y &= x \sqcap y \\ \mathit{not}\ x &= x \\ \mathit{minus}\ x\ y &= x \sqcap y \\ \mathit{times}\ x\ y &= x \sqcap y \end{aligned}$$

The Factorial Program, Revisited

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Start with $\bot_{2\to 2} = \{(\bot, \bot),\ (\top, \bot)\}$, the least fixed point iteration becomes

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We reach the least fixed point at $\{(\bot, \bot),\ (\top, \top)\}$. That is, *fac* is strict. When *fac* is applied to a non-terminating argument, it will diverge. When it is applied to others, it may or may not diverge.

Formalizing Abstract Interpretation

Let $abs \in D \to \bar{D}$ be an (intuitive) abstraction function that maps from a concrete domain D to an abstract domain $\bar{D}$. We can define on the Hoare power domain the abstraction and corresponding concretization functions

$$\begin{aligned} Abs &\in \mathrm{P}(D) \to \mathrm{P}(\bar{D}) \\ Conc &\in \mathrm{P}(\bar{D}) \to \mathrm{P}(D) \end{aligned}$$

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by

$$\begin{aligned} Abs\,(S) &= \mathrm{P}(abs)\,(S), \text{ where } \mathrm{P}(f)\,(X) = \{f(x) \mid x \in X\}^* \\ Conc\,(S) &= \bigcup \{T \mid Abs\,(T) \sqsubseteq_{\mathrm{P}(\bar{D})} S, T \in \mathrm{P}(D)\} \end{aligned}$$

Strictness Analysis, Revisited

For strictness analysis, it is straightforward to define $abs \in D \to 2$ for a concrete (basis) domain D as

$$abs\,(d) = \begin{cases} \bot_2 & \text{if } d = \bot_D \\ \top_2 & \text{if } d \neq \bot_D \end{cases}$$

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Then we have

$$Abs\,(X) = \begin{cases} \{\bot_2\} & \text{if } X = \{\bot_D\} \\ 2 & \text{otherwise} \end{cases}$$

$$Conc\,(X) = \begin{cases} \{\bot_D\} & \text{if } X = \{\bot_2\} \\ D & \text{if } X = 2 \end{cases}$$

Formalizing Abstract Interpretation, Continued

Let $f \in C \to D$ be a function, we define the abstraction function $abs \in (C \to D) \to (\bar{C} \to \bar{D})$ as

$$abs\,(f) \;=\; \bigsqcup \circ\; Abs \circ \mathrm{P}(f) \circ Conc \circ \{\cdot\}$$

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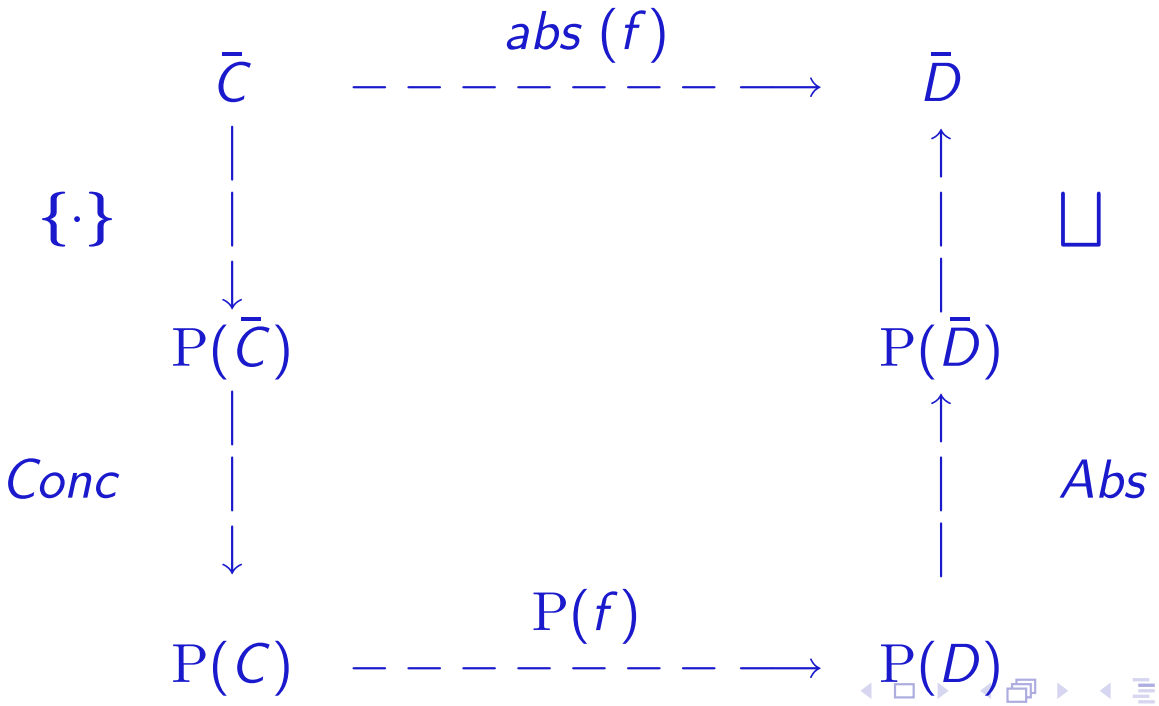

Formalizing Abstract Interpretation, Continued

This definition of abstraction for function is safe.

Theorem (Burn, Hankin, Abramsky)

Let function $f \in C \to D$. Then

$$\mathrm{P}(f) \sqsubseteq_{\mathrm{P}(C)\to\mathrm{P}(D)} \mathsf{Conc} \circ \mathrm{P}(\mathsf{abs}\,(f)) \circ \mathsf{Abs}$$

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This is illustrated by the following diagram:

$$\begin{array}{ccc} \mathrm{P}(C) & \xrightarrow{\quad \mathrm{P}(f) \quad} & \mathrm{P}(D) \\ \Big\downarrow \mathsf{Abs} & & \Big\uparrow \mathsf{Conc} \\ \mathrm{P}(\bar{C}) & \xrightarrow{\quad \mathrm{P}(\mathit{abs}\,(f)) \quad} & \mathrm{P}(\bar{D}) \end{array}$$

Strictness Analysis (Burn, Hankin, and Abramsky)

The following abstractions for built-in functions are safe.

1. If f is strict in all of its n arguments, then define

$$(abs\,(f))\ x_1\ x_2\ \ldots\ x_n\ =\ x_1 \sqcap x_2 \sqcap \ldots \sqcap x_n$$

2. Let $\mathit{if\text{-}then\text{-}else} \in \mathcal{B} \to D \to D \to D$ be the standard semantics of the "if then else" construct. Then define

$$(abs\,(\mathit{if\text{-}then\text{-}else}))\ x\ y\ z\ =\ x\ \mathbf{and}\ (y \sqcup z),$$

where $\mathbf{and} \in 2 \to \bar{D} \to \bar{D}$ is defined by

$$\begin{aligned} \bot\ \mathbf{and}\ e &= \bot_{\bar{D}} \\ \top\ \mathbf{and}\ e &= e \end{aligned}$$

3. If $f \in D \to D$, then define

$$abs\,(fix\,(f)) = fix\,(abs\,(f))$$