# Denotational Semantics

Homework

Due 9:30 am, July 9, 2010

Recall the function `fib` to compute the Fibonacci numbers, now written in Objective Caml (O'Caml):

```
let rec fib n = if (n=0) or (n=1) then 1 else fib (n-1) + fib (n-2)
```

We have that `fib 0` computes to 1; `fib 1` computes to 1; `fib 2` computes 2; `fib 3` computes 3; etc. Notice that, however, if the argument `n` is non-terminating, then the computation `fib n` will not terminate as well.

Based on the above function `fib`, we write a non-recursive function `f` below:

```
let f fib n = if (n=0) or (n=1) then 1 else fib (n-1) + fib (n-2)
```

Note that O'Caml infers the type of `f` to be `(int -> int) -> (int -> int)`. That is, given a function `fib` of type `int -> int`, the result of function application `f fib` is also a function of type `int -> int`.

For the purpose of simplification, we assume we are working on natural numbers instead of integers. Therefore, it is understood that function `f` has a meaning $f$, where $f \in (\mathcal{N} \to \mathcal{N}) \to (\mathcal{N} \to \mathcal{N})$, and the meaning of function `fib` is the least fixed point of $f$. From the definition of function `f`, we derive the definition of $f$ as

$$f\,\mathit{fib}\;n \;=\; \mathit{if\text{-}then\text{-}else}\;(\mathit{or}\;(\mathit{eq}\;n\;0)\;(\mathit{eq}\;n\;1))\;1\;(\mathit{plus}\;(\mathit{fib}\;(\mathit{minus}\;n\;1))\;(\mathit{fib}\;(\mathit{minus}\;n\;2)))$$

or, equivalently,

$$f\,\mathit{fib} \;=\; \{(n, \mathit{if\text{-}then\text{-}else}\;(\mathit{or}\;(\mathit{eq}\;n\;0)\;(\mathit{eq}\;n\;1))\;1\;(\mathit{plus}\;(\mathit{fib}\;(\mathit{minus}\;n\;1))\;(\mathit{fib}\;(\mathit{minus}\;n\;2)))) \mid n \in \mathcal{N}\}$$

where functions $\mathit{if\text{-}then\text{-}else} \in \mathcal{B} \to \mathcal{N} \to \mathcal{N} \to \mathcal{N}$, $\mathit{or} \in \mathcal{B} \to \mathcal{B} \to \mathcal{B}$, $\mathit{eq} \in \mathcal{N} \to \mathcal{N} \to \mathcal{B}$, and $\mathit{plus}, \mathit{minus} \in \mathcal{N} \to \mathcal{N} \to \mathcal{N}$ are defined as usual.

Start from $\bot_{\mathcal{N}\to\mathcal{N}} = \{(n, \bot) \mid n \in \mathcal{N}\}$, we can compute

$$\begin{array}{rcl}
f^{(0)}\;(\bot_{\mathcal{N}\to\mathcal{N}}) &=& \bot_{\mathcal{N}\to\mathcal{N}} \\
f^{(1)}\;(\bot_{\mathcal{N}\to\mathcal{N}}) &=& f(f^{(0)}(\bot_{\mathcal{N}\to\mathcal{N}})) \\
f^{(2)}\;(\bot_{\mathcal{N}\to\mathcal{N}}) &=& f(f^{(1)}(\bot_{\mathcal{N}\to\mathcal{N}})) \\
f^{(3)}\;(\bot_{\mathcal{N}\to\mathcal{N}}) &=& f(f^{(2)}(\bot_{\mathcal{N}\to\mathcal{N}})) \\
& \cdots & \\
f^{(k+1)}\;(\bot_{\mathcal{N}\to\mathcal{N}}) &=& f(f^{(k)}(\bot_{\mathcal{N}\to\mathcal{N}})) \\
& \cdots &
\end{array}$$

1. What are $f^{(0)}\;(\bot_{\mathcal{N}\to\mathcal{N}})$, $f^{(1)}\;(\bot_{\mathcal{N}\to\mathcal{N}})$, $f^{(2)}\;(\bot_{\mathcal{N}\to\mathcal{N}})$, and $f^{(3)}\;(\bot_{\mathcal{N}\to\mathcal{N}})$?
   (Hint: They are functions. You can express them in the form of

   $$\{(0, ?), (1, ?), (2, ?), \ldots (k, ?)\} \cup \{(n, \bot) \mid n \in \mathcal{N} - \{0, 1, \ldots, k\}\}$$

   Just fill in the ?, and be precise about $k$.)

2. What is the least upper bound of the following set?

   $$\{f^{(0)}\;(\bot_{\mathcal{N}\to\mathcal{N}}),\; f^{(1)}\;(\bot_{\mathcal{N}\to\mathcal{N}}),\; f^{(2)}\;(\bot_{\mathcal{N}\to\mathcal{N}}),\; \ldots, f^{(k+1)}\;(\bot_{\mathcal{N}\to\mathcal{N}}),\; \ldots\}$$

   Describe your answer informally but justify it.