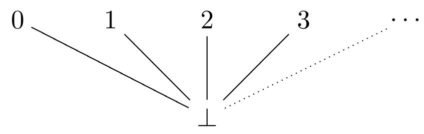
Denotational Semantics

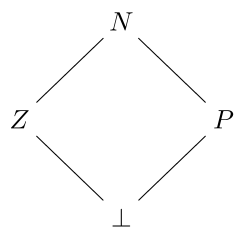
Another In–class Exercise

July 8, 2010

The following is the concrete domain $\mathcal{N}$ for natural numbers



The following is the abstract domain $\bar{\mathcal{N}}$ for (non-)zero property of elements in $\mathcal{N}$



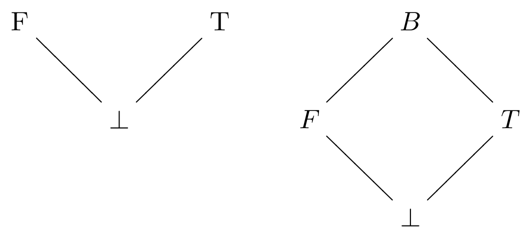
with the obvious abstraction $abs_{\mathcal{N}\to\bar{\mathcal{N}}}$

$$abs\ x = \begin{cases} \bot & \text{if } x = \bot \\ Z & \text{if } x = 0 \\ P & \text{otherwise} \end{cases}$$

Note that $\bar{\mathcal{N}}$ is a complete lattice. How do the Hoare power domains $\mathrm{P}(\mathcal{N})$ and $\mathrm{P}(\bar{\mathcal{N}})$ look like? How do functions $Abs \in \mathrm{P}(\mathcal{N}) \to \mathrm{P}(\bar{\mathcal{N}})$ and $Conc \in \mathrm{P}(\bar{\mathcal{N}}) \to \mathrm{P}(\mathcal{N})$ look like? Please complete the following equation:

$$Conc\ (X) = \begin{cases} \underline{\hspace{3cm}} & \text{if } X = \{\bot\} \\ \underline{\hspace{3cm}} & \text{if } X = \{\bot, Z\} \\ \underline{\hspace{3cm}} & \text{if } X = \{\bot, P\} \\ \underline{\hspace{3cm}} & \text{if } X = \{\bot, Z, P, N\} \end{cases}$$

For our analysis below, we also make use of the concrete domain $\mathcal{B}$ for booleans, and the obvious abstraction domain $\bar{\mathcal{B}}$



Now, look at the following Objective Caml code for *gcd* for *natural numbers*:

```
let rec gcd (m, n) =
     if m mod n = 0
        then n else gcd (n, m mod n)
```

How do you perform a (non-)zero analysis of this program? That is, how do you compute a function $\bar{gcd} \in \bar{\mathcal{N}} \times \bar{\mathcal{N}} \to \bar{\mathcal{N}}$ so that it is a safe abstract interpretation of *gcd* in terms of of its (non-)zero property?

You can start with a safe abstraction of the equality function and the *mod* function. The following is the table for $\bar{eq}$:

$\bar{eq}\ (x,y)$	$y=\bot$	$y=Z$	$y=P$	$y=N$
$x=\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$x=Z$	$\bot$	T	F	B
$x=P$	$\bot$	F	B	B
$x=N$	$\bot$	B	B	B

What are the tables for $\bar{mod}$ and $\bot_{\bar{\mathcal{N}}\times\bar{\mathcal{N}}\to\bar{\mathcal{N}}}$?

$\bar{mod}\ (x,y)$	$y=\bot$	$y=Z$	$y=P$	$y=N$
$x=\bot$				
$x=Z$				
$x=P$				
$x=N$				

Use the above tables to do a least fixed point iteration for computing $\bar{gcd}$.