

Two-Player Perfect Information Games: A Brief Survey

Tsan-sheng Hsu

徐讚昇

tshsu@iis.sinica.edu.tw

<http://www.iis.sinica.edu.tw/~tshsu>

Abstract

- Domain: two-player games.
- Which game characters are predominant when the solution of a game is the main target?
 - It is concluded that **decision complexity** is more important than **state-space complexity**.
 - There is a trade-off between **knowledge-based methods** and **brute-force methods**.
 - There is a clear correlation between the first-player's **initiative** and the necessary effort to solve a game.

Definitions (1/4)

- **Domain: 2-person zero-sum games with perfect information.**
 - **Zero-sum** means one player's loss is exactly the other player's gain, and vice versa.
 - ▷ *There is no way for both players to win at the same time.*
- **Game-theoretic value** of a game: the outcome, i.e., win, loss or draw, when all participants play optimally.
 - Classification of games' solutions according to L.V. Allis [Ph.D. thesis 1994] if they are considered solved.
 - ▷ *Ultra-weakly solved: the game-theoretic value of the initial position has been determined.*
 - ▷ *Weakly solved: for the initial position a strategy has been determined to achieve the game-theoretic value against any opponent.*
 - ▷ *Strongly solved: a strategy has been determined for all legal positions.*
 - The game-theoretical values of most games are **unknown** or are only known for some **legal positions**.
 - ▷ *A legal position is one that can be reached from the initial position.*

Definitions (2/4)

- **State-space** complexity of a game: the number of the legal positions in a game.
- **Game-tree** (or **decision**) complexity of a game: the number of the nodes in a **solution search tree**.
 - A solution search tree is a tree where the game-theoretic value of the root position can be decided.
 - Each node in the tree is a legal position. The children of a parent node P are the positions that P can reach in one steps.
 - Some legal states may not be in a solution search tree.
 - ▷ *These are unreasonable positions.*
 - Some children of a node may not be in a solution search tree.
- A **fair** game: the game-theoretic value is draw and both players have roughly an equal probability on making a mistake.
 - ▷ *Paper-scissor-stone*
 - ▷ *Roll a dice and compare who gets a larger number*
- **Initiative**: the right to move first.

Definitions (3/4)

- A **convergent** game: the size of the state space decreases as the game progresses.
 - Start with many pieces on the board and pieces are gradually removed during the course of the game.
 - ▷ *Example: Checkers.*
 - It means the number of possible configurations decreases as the game progresses.
- A **divergent** game: the size of the state space increases as the game progresses.
 - May start with an empty board, and pieces are gradually added during the course of the game.
 - ▷ *Example: Connect-5 before the board is almost filled.*
 - It means the number of possible configurations increases as the game progresses.

Definitions (4/4)

- A game may be convergent at one stage and then divergent at other stage.
 - Most games are dynamic.
 - For the game of Tic-Tac-Toe, assume you have played x plys with x being even.
 - ▷ *Then you have a possible of*

$$\begin{pmatrix} 9 \\ x/2 \end{pmatrix} \begin{pmatrix} 9 - x/2 \\ x/2 \end{pmatrix}$$

different configurations.

- This number is not monotone increasing or decreasing.

Predictions made in 1990

- Predictions were made in 1990 [Allis et al 1991] for the year 2000 concerning the expected playing strength of computer programs.

solved	over champion	world champion	grand master	amateur
Connect-four	Checkers (8 * 8)	Chess	Go (9 * 9)	Go (19 * 19)
Qubic	Renju	Draughts (10 * 10)	Chinese chess	
Nine Men's Morris	Othello		Bridge	
Go-Moku	Scrabble			
Awari	Backgammon			

- ▷ *Over champion means definitely over the best human player.*
- ▷ *World champion means equaling to the best human player.*
- ▷ *Grand master means beating most human players.*

A double dichotomy of the game space

$\log \log(\text{state-space complexity}) \uparrow$

category 3 if solvable at all, then by knowledge-based methods	category 4 currently unsolvable by any method
category 1 solvable by any method	category 2 if solvable at all, then by brute-force methods

$\log \log(\text{game-tree complexity}) \rightarrow$

Questions to be researched

- Can perfect knowledge obtained from solved games be translated into rules and strategies which human beings can assimilate?
- Are such rules generic, or do they constitute a multitude of ad hoc recipes?
- Can methods be transferred between games?
 - More specifically, are there generic methods for all category- n games, or is each game in a specific category a law unto itself?

Convergent games

- Since most games are dynamic, here we consider games whose ending phases are convergent.
 - Can be solved by the method of **endgame databases** if we can enumerate and store all possible positions at a certain stage.
- Problems solved:
 - Nine Men's Morris: in the year 1995, a total of 7,673,759,269 states.
 - ▷ *The game theoretic value is draw.*
 - Mancala games
 - ▷ *Awari: in the year 2002.*
 - ▷ *Kalah: in the year 2000 upto, but not equal, Kalah(6,6)*
 - Checkers
 - ▷ *By combining endgame databases, middle-game databases and verification of opening-game analysis.*
 - ▷ *Solved the so called 100-year position in 1994.*
 - ▷ *The game is proved to be a draw in 2007.*
 - Chess endgames
 - Chinese chess endgames

Divergent games

- Since most games are dynamic, here we consider games whose INITIAL phases are divergent.
- Connection games
 - Connect-four ($6 * 7$)
 - Qubic ($4 * 4 * 4$)
 - Go-Moku ($15 * 15$)
 - Renju
 - k -in-a-row games
 - Hex ($10 * 10$ or $11 * 11$)
- Polynmino games: place pieces inside a board without overlapping and alternatively until one cannot place more.
 - Pentominoes
 - Domineering
- Othello
- Chess
- Chinese chess
- Shogi
- Go

Connection games (1/2)

■ Connect-four ($6 * 7$)

- Solved by J. Allen in 1989 using a brute-force depth first search with alpha-beta pruning, a transposition table, and killer-move heuristics.
- Also solved by L.V. Allis in 1988 using a knowledge-based approach by combining 9 strategic rules that identify potential **threats** of the opponent.
 - ▷ *Threats are something like forced moves or moves you have little choices.*
 - ▷ *Threats are moves with predictable counter-moves.*
- It is first-player win.
- Weakly solved on a SUN-4 workstation using 300+ hours.

■ Qubic ($4 * 4 * 4$)

- A three-dimensional version of Tic-Tac-Toe.
- Connect-four played on a $4 * 4 * 4$ game board.
- Solved in 1980 by O. Patashnik by combining the usual depth-first search with expert knowledge for ordering the moves.
 - ▷ *It is first-player win for the 2-player version.*

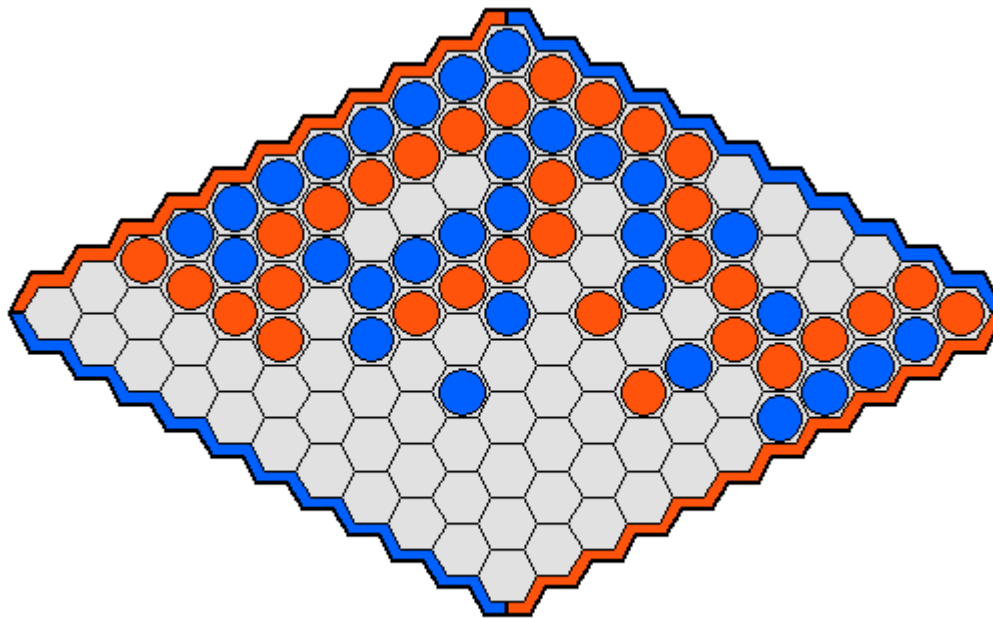
Connection games (2/2)

- **Go-Moku (15 * 15)**
 - First-player win.
 - Weakly solved by L.V. Allis in 1995 using a combination of threat-space search and database construction.
- **Renju**
 - Does not allow the first player to play certain moves.
 - An *asymmetric* game.
 - Weakly solved by Wágner and Virág in 2000 by combining search and knowledge.
 - ▷ *Took advantage of an iterative-deepening search based on threat sequences up to 17 plies.*
 - ▷ *It is still first-player win.*
- **k -in-a-row games**
 - mnk -Game: a game playing on a board of m rows and n columns with the goal of obtaining a straight line of length k .
 - Variations: first ply picks only one stone, the rest picks two stones in a ply.
 - ▷ *Connect 6.*
 - ▷ *Try to balance the advantage of the initiative!*

Hex (10 * 10 or 11 * 11)

■ Properties:

- It is a finite game.
- It is not possible for both players to win at the same time.
- Exactly one of the players can win.



Red won

Courtesy of ICGA web site

Proof on exactly one player win (1/2)

- It is easy to know there cannot be two winners.
 - ▷ *When the first player wins, allow the second player to play one more time. If the second player also wins, then the game is tie.*
- A topological argument.
 - A vertical chain can only be cut by a horizontal chain and vice versa because each cell is connected with 6 adjacent cells.
 - ▷ *Note if a cell has 4 neighbors as in the case of Go, then it is possible to cut off a vertical chain by cells that are not horizontally connected and vice versa.*
- Other arguments such as one using graph theory exist.

Proof on exactly one player win (2/2)

- We then prove there is at least one winner.
 - Assume there is no winner.
- W.l.o.g. let R be the set of red cells that can be reached by chains originated from the rightmost column.
 - R must contain a cell of the leftmost column; otherwise we have a contradiction.
- Let $N(R)$ be the blue cells that can be reached by R originated from the rightmost column.
 - $N(R)$ must contain a cell in the top row.
 - ▷ *Otherwise, R contains all cells in the first row, which is a contradiction.*
 - $N(R)$ must contain a cell in the bottom row.
 - ▷ *Otherwise, R contains all cells in the bottom row, which is a contradiction.*
 - $N(R)$ must be connected.
 - ▷ *Otherwise, R can advance further.*
 - Hence $N(R)$ is a blue winning chain.

Illustration of the ideas (1/3)

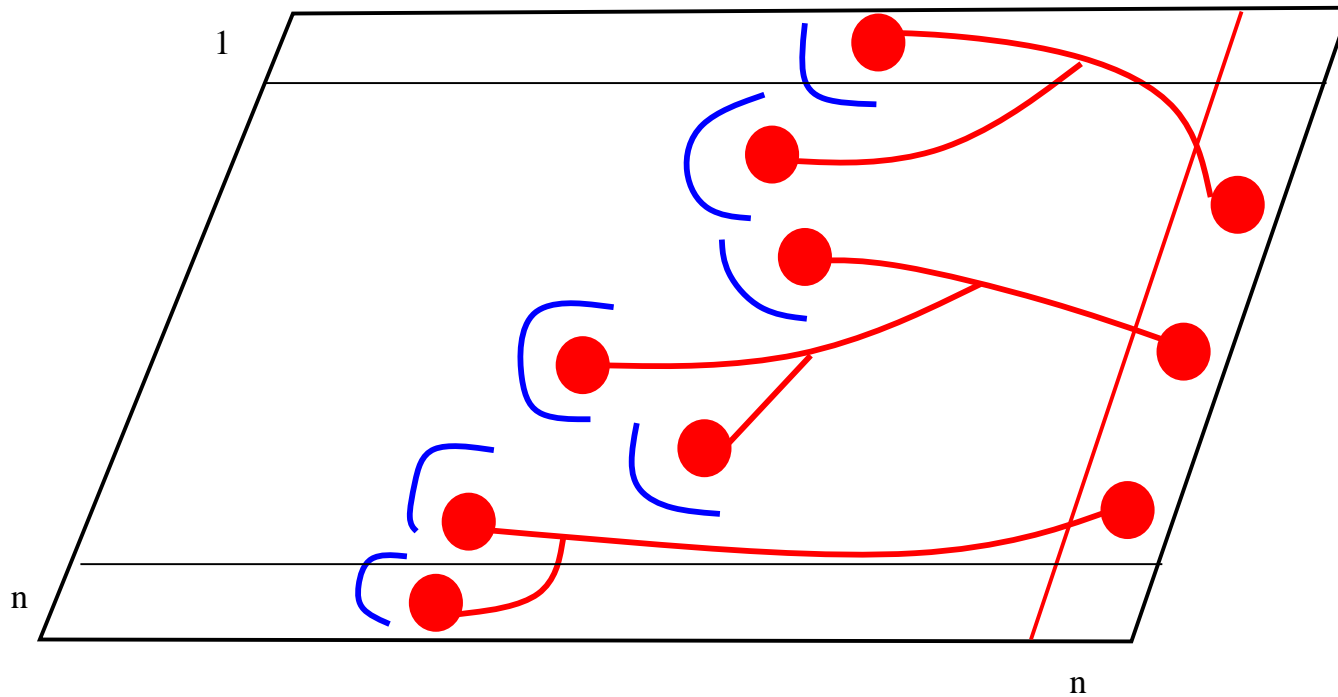


Illustration of the ideas (2/3)

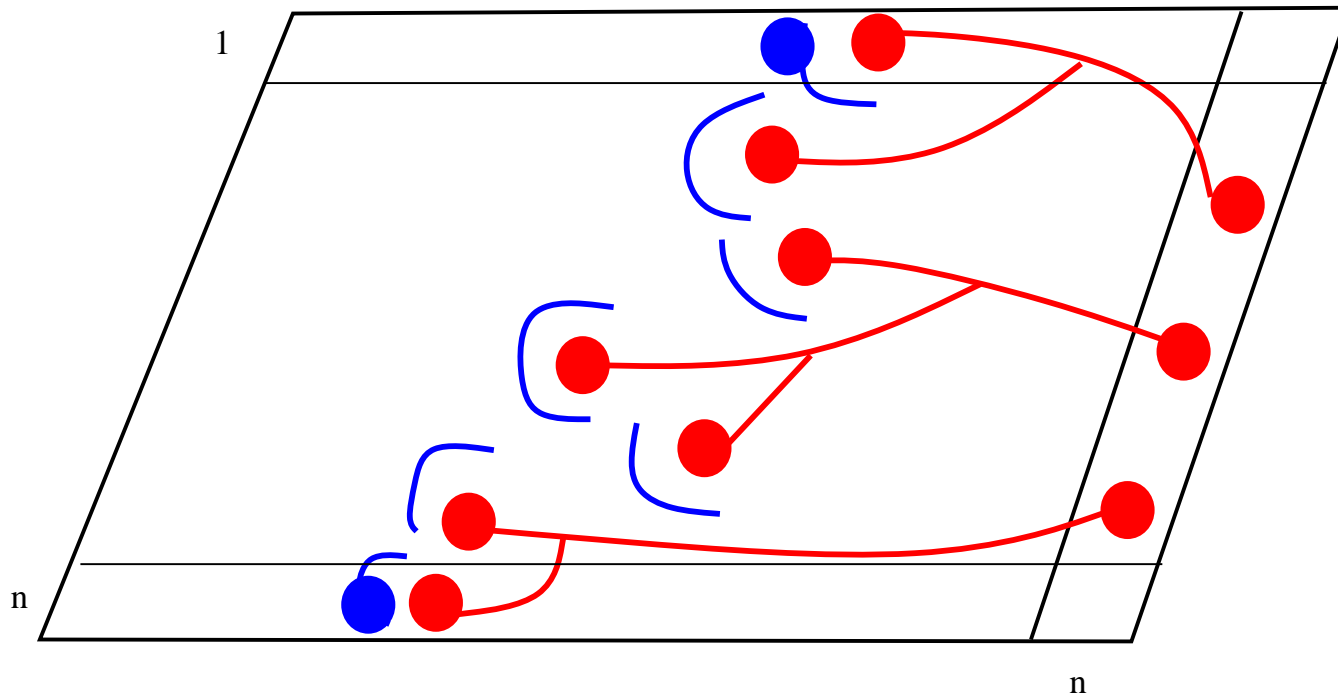
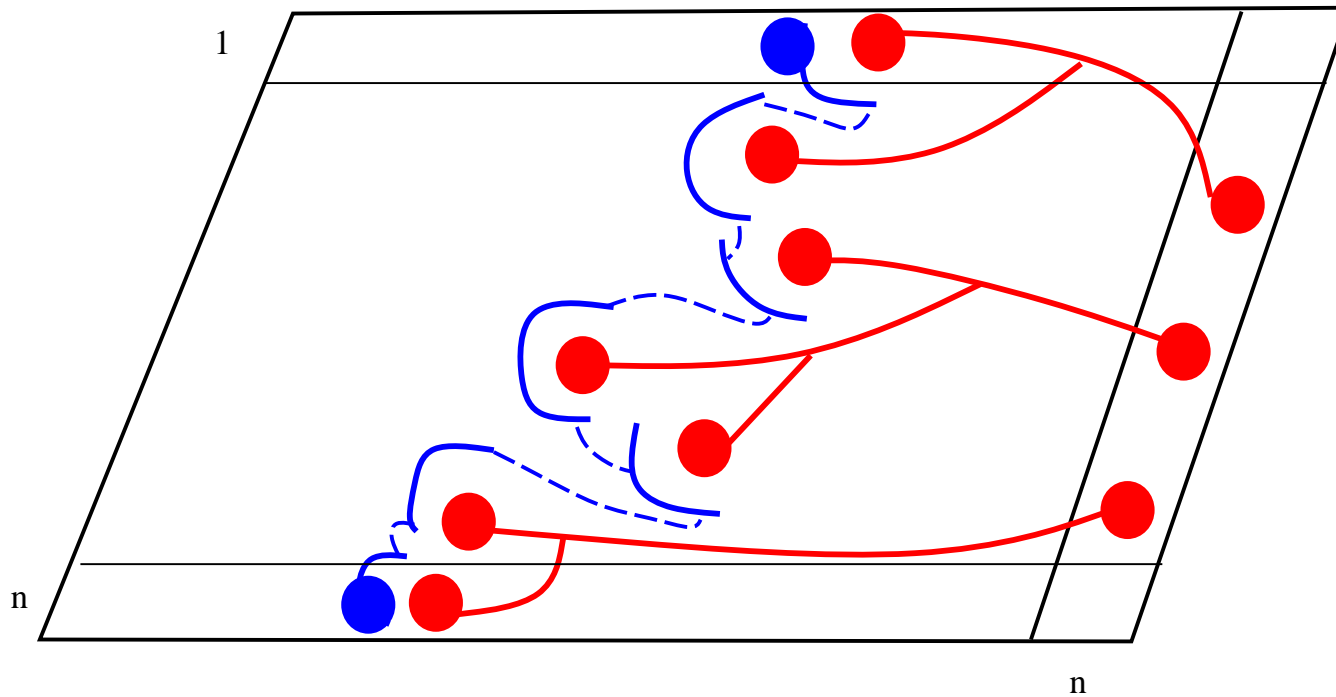


Illustration of the ideas (3/3)



Strategy-stealing argument (1/4)

- *The unrestricted form of Hex is a first-player win game. using the “**strategy-stealing**” argument made by John Nash in 1949.*
 - If there is a winning strategy for the second player, the first player can still win by making an arbitrary first move and using the second-player strategy from then on.
 - ▷ *The first player ignores the arbitrary first move by assuming that move does not exist.*
 - ▷ *Hence the second move made by the second player becomes the first move.*
 - ▷ *The third move made by the first player becomes the second move.*
 - If using the second-player strategy requires playing the chosen first move or any move played before, then make another arbitrary move.
 - ▷ *An arbitrary extra move can never be a disadvantage in Hex.*
 - We have obtained a contradiction, and thus the second player cannot win.
 - Since we have proved there is no draw, and there is always a winner, and both players cannot win at the same time, the first player must have a winning strategy.

Strategy-stealing argument (2/4)

- Assume the second player P_2 has a winning function $f(B)$ that tells the next ply towards winning when seeing the board B .
 - Assume the initial board position is B_0 .
 - $f(B)$ has a value only when B is a legal position for the second player.
 - $rev(x)$: interchange colors of pieces in a board or ply x .
- The steps taken by the first player P_1 to also win
 - P_1 makes an arbitrary first ply m_1 . Call it m' .
 - P_2 uses $f(B_0 + m_1)$ to make the second ply m_2 .
 - P_1 makes the third ply $m_3 = rev(f(B_0 + rev(m_2)))$.
 - ▷ If $m_3 = m'$, then make another arbitrary ply and let it be the new m' .
 - P_2 uses $f(B_0 + m_1 + m_2 + m_3)$ to make the forth ply m_4 .
 - P_1 makes the fifth ply $m_5 = rev(f(B_0 + rev(m_2) + m_3 + rev(m_4)))$.
 - ▷ If $m_5 = m'$ or any ply made before, then make another arbitrary ply and let it be the new m' .
 - ...

Strategy-stealing argument (3/4)

- Hence we know it is not possible for the second player to win.
- We also know these.
 - There is exactly one winner when the board is completely filled.
 - The game is finite.
 - Hence we can enumerate the whole solution search tree.
 - ▷ *In this solution search tree, there is a way for one player to win all of the times no matter the opponent reacts.*
- Since the second player cannot win, the first player must have a winning strategy.

Strategy-stealing argument (4/4)

- This is not a constructive proof.
- The strategy-stealing argument cannot be used for every game.
 - An arbitrary extra move can never be a disadvantage in Hex.
 - This may not be true for other games.
- The argument works for any game when
 - it is symmetric,
 - it is history independent,
 - it always has exactly one winner, and
 - ▷ *namely, it cannot have a draw by having no winners or 2 winners,*
 - an arbitrary extra move can never be a disadvantage.
 - ▷ *Note: it requires that a player is always possible to place an arbitrary move which may not be true for some games.*

Properties of Hex

■ Variations of Hex

- The **one-move-equalization** rule: one player plays an opening move and the other player then has to decide which color to play for the remainder of the game.
 - ▷ *The revised version is a second-player win game (ultra-weakly).*

■ Hex exhibits considerable mathematical structure.

- Hex in its general form has been proved to be PSPACE-complete by Even and Tarjan in 1976 by converting it to a Shannon switching game.
- The state-space and decision complexities are comparable to those of Go on an **equally-sized** board.

■ Solutions

- (Weakly or strongly) solved on a $6 * 6$ board in 1994.
- Maybe possible to solve the $7 * 7$ case.
 - ▷ *The $7 * 7$ case was solved in 2001. [Yang et. al. 2001]*
- Not likely to solve the $8 * 8$ version without fundamental breakthroughs.
 - ▷ *The $8 * 8$ case was solved in 2009. [Henderson et. al. 2009]*

More divergent games (1/3)

- **Polynmino games: placing 2-D pieces of a connected subset of a square grid to construct a special form.**
 - Pentominoes
 - Domineering
 - Games on smaller boards have been solved.
- **Othello**
 - M. Buro's LOGISTELLO beat the resigning World Champion by 6-0 in 1997.
 - Weakly solved on a $6 * 6$ board by J. Feinstein in 1993.
 - ▷ *Second player win.*
- **Chess**
 - DEEP BLUE beat the human World Champion in 1997!

More divergent games (2/3)

■ Chinese chess

- Still in progress.
- Professional 7-dan since 2007.

■ Shogi

- Still in progress.
- Claimed to be professional 2-dan in 2007.
- Defeat a Lady professional player in 2010.
- Defeat a 68-year old 1993 Meijin during 2011 and 2012.

More divergent games (3/3)

■ Go

- Recent success and breakthrough using Monte Carlo UCT based methods between 2004 and 2012.
- Lack major theoretical or practical break through since 2012.
- Amateur 1 – 4 kyu in 2008.
 - ▷ *Beat a professional 8-dan by having an 8-stone advantage.*
 - ▷ *Beaten by a professional 9-dan by giving a 7-stone advantage.*
- Amateur 1 dan in 2010.
- Amateur 3 dan in 2011.
- The program Zen beat a 9-dan professional master at March 17, 2012.
 - ▷ *First game: Five stone handicap and won by 11 points.*
 - ▷ *Second game: four stones handicap and won by 20 points.*

Table of complexity

Game	$\log_{10}(\text{state-space})$	$\log_{10}(\text{game-tree size})$
Nine Men's Morris	10	50
Pentominoes	12	18
Awari	12	32
Kalak(6,4)	13	18
Connect-four	14	21
Domineering (8 * 8)	15	27
Dakon-6	15	33
Checkers	21	31
Othello	28	58
Qubic	30	34
Draughts	30	54
Chess	46	123
Chinese chess	48	150
Hex (11 * 11)	57	98
Shogi	71	226
Renju (15 * 15)	105	70
Go-Moku (15 * 15)	105	70
Go (19 * 19)	172	360

State-space versus game-tree size

- In 1994, the boundary of solvability by complete enumeration was set at 10^{11} .
 - The current estimation is about 10^{13} (since the year 2007).
- It is often possible to use heuristics in searching a game tree to cut the number of nodes visited tremendously when the structure of the game is well studied.
 - Example: Connect-Four.

Methods developed for solving games

■ Brute-force methods

- Retrograde analysis
- Enhanced transposition-table methods

■ Knowledge-based methods

- Threat-space search and λ -search
- Proof-number search
- Depth-first proof-number search
- Pattern search

- ▷ To search for *threat patterns*, which are collections of cells in a position.
- ▷ A threat pattern can be thought of as representing the *relevant area* on the board, an area that human players commonly identify when analyzing a position.

■ Recent advancements:

- Monte Carlo UCT based game tree simulation.
 - ▷ Monte Carlo method has a root from statistic.
 - ▷ Biased sampling.
 - ▷ Using methods from machine learning.
 - ▷ Combining domain knowledge with statistics.
- A majority vote algorithm.

Brute-force versus knowledge-based methods

- Games with both a relative low state-space complexity and a low game-tree complexity have been solved by both methods.
 - **Category 1**
 - Connect-four and Qubic
- Games with a relative low state-space complexity have mainly been solved with brute-force methods.
 - **Category 2**
 - Namely by constructing endgame databases
 - Nine Men's Morris
- Games with a relative low game-tree-complexities have mainly been solved with knowledge-based methods.
 - **Category 3**
 - Namely, by intelligent (heuristic) searching
 - Sometimes, with the helps of endgame databases
 - Go-Moku, Renju, and k -in-a-row games

Advantage of the initiative

- **Theorem (or argument) made by Singmaster in 1981: The first player has advantages.**
 - **Two kinds of positions**
 - ▷ *P-positions: the previous player can force a win.*
 - ▷ *N-positions: the next player can force a win.*
 - **Arguments**
 - ▷ *For the first player to have a forced win, just one of the moves must lead to a P-position.*
 - ▷ *For the second player to have a forced win, all of the moves must lead to N-positions.*
 - ▷ *It is easier to the first player to have a forced win assuming all positions are randomly distributed.*
 - ▷ *Can be easily extended to games with draws.*
- **Remarks:**
 - **One small boards, the second player is able to draw or even to win for certain games.**
 - **Cannot be applied to the infinite board.**

How to make use of the initiative

- **A potential universal strategy for winning a game:**
 - Try to obtain a small advantage by using the initiative.
 - ▷ *The opponent must react adequately on the moves played by the other player.*
 - To reinforce the initiative the player searches for threats, and even a sequence of threats using an evaluation function E .
 - Force the opponent to always play the moves you expected.
- **Threat-space search**
 - Search for threats only!

Offsetting the initiative

- An example of a game with a huge initiative:
 - A connection $mn1$ -game.
 - ▷ 一子棋 was mentioned in 張系國著名小說”棋王”(1978年出版).
 - A connection $mn2$ -game.
 - A connection $mn3$ -game.
- Need to offset the initiative.
 - The offsetting rule must be simple.
 - The revised game must be as **fair** as possible.
 - ▷ *It is difficult to prove a game is fair.*
 - ▷ *Example: Paper-scissor-stone is fair.*
 - The revised game needs be fun to play with.
 - The revised game cannot be too much different from the original game.
- Knowing how to properly offsetting the initiative may uncover some fundamental properties of the game such as its level of difficulty.

Examples (1/2)

- Enforce rules so that the first player cannot win by selective patterns.
 - Renju.
 - ▷ *Still first-player win.*
 - Go (19 * 19).
 - ▷ *The first player must win by more than 7 stones.*
 - ▷ *Komi = 7.5 in 2011.*
 - ▷ *The value of Komi changes with the time and maybe different because of using different set of rules.*
- The **one-move-equalization** rule: one player plays an opening move and the other player then has to decide which color to play for the remainder of the game.
 - ▷ *Hex.*
 - ▷ *Second-player win.*

Examples (2/2)

- **The first move plays one stone, the rest plays two stones each.**
 - ▷ *Connect 6.*
 - ▷ *Intuitively, in each turn the initiative goes to different players alternatively.*
 - ▷ *Still not able to prove the game is fair (at 2015).*
- **The first player uses less resource.**
 - For example: using less time.
 - ▷ *Chinese chess.*
 - A resource-auctioning scheme.
- **Unclear about how to redesign a game to make it fair.**

Conclusions

- **The knowledge-based methods mostly inform us on the structure of the game, while exhaustive enumeration rarely does.**
- **Many ad-hoc recipes are produced currently.**
 - The database can be used as a corrector or verifier of strategies formulated by human experts.
- **It may be hopeful to use data mining techniques to obtain cross-game methods.**
 - Currently not very successful.

Comments

- Can combine knowledge-based method with exhaustive enumeration.
 - For converging games, build endgame databases when the remaining state spaces is manageable.
 - ▷ *Example: build endgames with at most 5 pieces in Chess and stop searching when the number of pieces on the board is less than 6.*
 - For diverging games, pre-compute all possible opening moves and solve them one by one in sequence or in parallel.
- This is different from the usage of pattern databases in solving one-player games.
 - Patterns are used to guide the search in solving one-player games.
 - Endgame databases are used here to stop the search earlier. The idea has a flavor like that of bi-directional search.

1990's Predictions — 2000's Status

- Predictions were made in 1990 [Allis et al 1991] for the year 2000 concerning the expected playing strength of computer programs.

solved	over champion	world champion	grand master	amateur
Connect-four	Checkers (8 * 8)	Chess	Go (9 * 9)	Go (19 * 19)
Qubic	Renju	Draughts (10 * 10)	Chinese chess	
Nine Men's Morris	Othello		Bridge	
Go-Moku	Scrabble			
Awari	Backgammon			

- **color code**

- **Green:** Performs much better than expected
- **Red:** right on the target.
- **Black:** have some progress towards the target.
- **Blue:** not so.

Predictions for 2010

- Predictions were made at the year 2000 for the year 2010 concerning the expected playing strength of computer programs.

solved	over champion	world champion	grand master	amateur
Awari	Chess	Go (9 * 9)	Bridge	Go (19 * 19)
Othello	Draughts (10 * 10)	Chinese chess	Shogi	
Checkers (8 * 8)	Scrabble	Hex		
	Backgammon	Amazons		
	Lines of Action			

Predictions for 2010 – Status

- My personal opinion about the status of Prediction-2010 at October, 2010, right after the Computer Olympiad held in Kanazawa, Japan.

solved	over champion	world champion	grand master	amateur
Awari	Chess	Go (9 * 9)	Bridge	Go (19 * 19)
Othello	Draughts (10 * 10)	Chinese chess	Shogi	
Checkers (8 * 8)	Scrabble	Hex		
	Backgammon	Amazons		
	Lines of Action			

- color code
 - **Red**: right on the target.
 - **Black**: have some progress towards the target.
 - **Blue**: not so.

References and further readings (1/2)

- L.V. Allis, H.J. van den Herik, and I.S. Herschberg. Which games will survive? In: D.N.L. Levy, D.F. Beal (Eds.), *Heuristic Programming in Artificial Intelligence 2: The Second Computer Olympiad*, Ellis Horwood, Chichester, 1991, pp. 232-243.
- * H. J. van den Herik, J. W. H. M. Uiterwijk, and J. van Rijswijck. Games solved: Now and in the future. *Artificial Intelligence*, 134:277–311, 2002.
- Jonathan Schaeffer. The games computers (and people) play. *Advances in Computers*, 52:190–268, 2000.
- L. V. Allis, M. van der Meulen, and H. J. van den Herik. Proof-number search. *Artificial Intelligence*, 66(1):91–124, 1994.

References and further readings (2/2)

- J. Yang, S. Liao, and M. Pawlak. A decomposition method for finding solution in game Hex 7x7. In *Proceedings of International Conference on Application and Development of Computer games in the 21st century*, pages 93–112, November 2001.
- P. Henderson, B. Arneson, and R. B. Hayward. Solving 8x8 Hex. In *Proceedings of IJCAI*, pages 505–510, 2009.