Heuristic Search with Pre-Computed Databases

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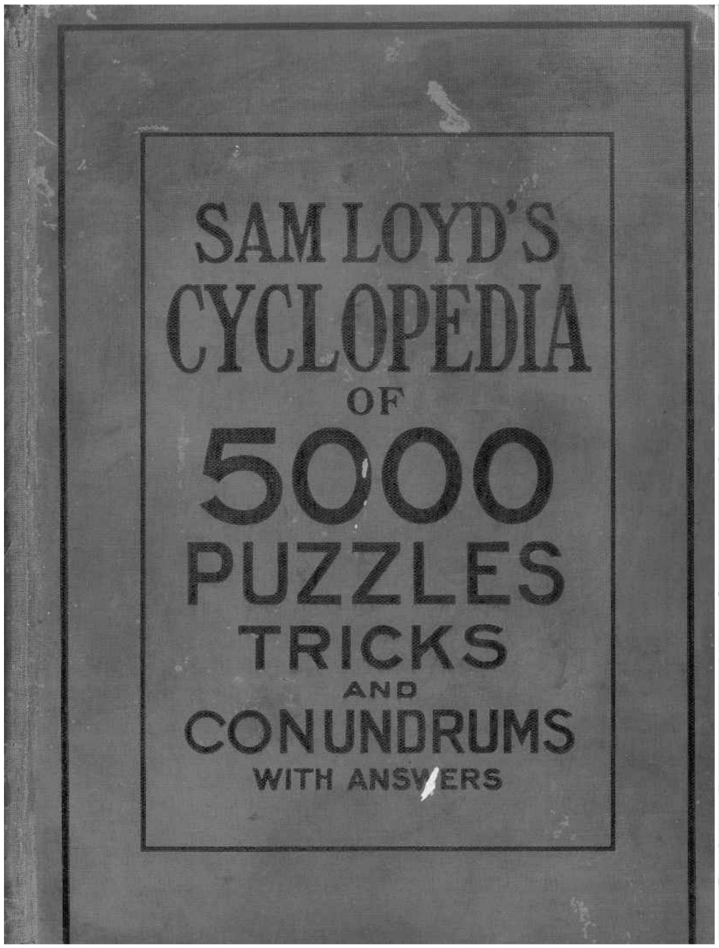
Abstract

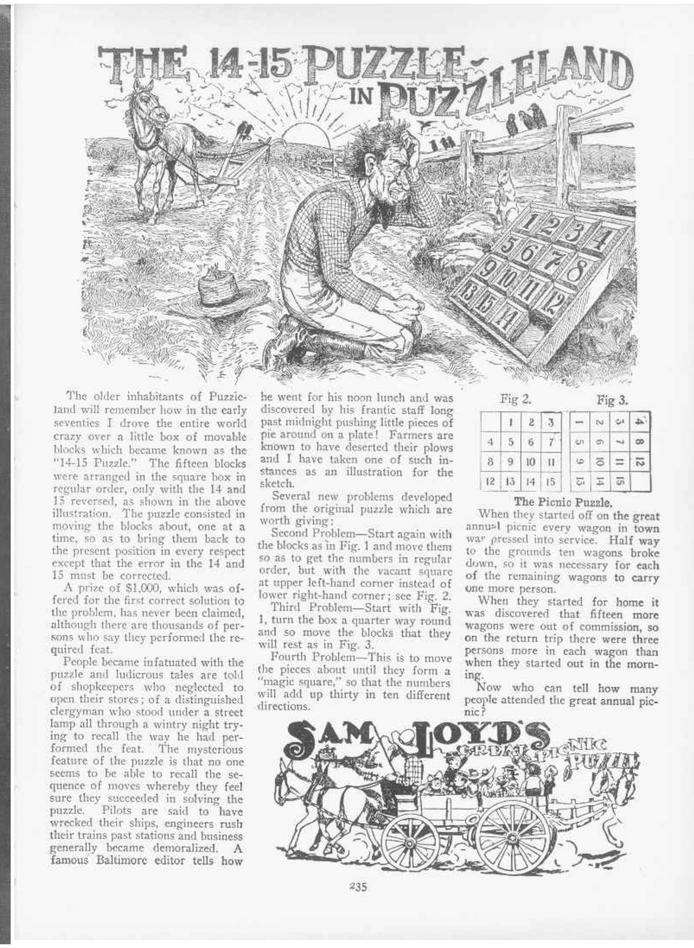
- Use pre-computed partial results to improve the efficiency of heuristic search.
- Introducing a new form of heuristic called pattern databases.
  - Compute the cost of solving individual subgoals independently.
  - If the subgoals are disjoint, then we can use the sum of costs of the subgoals as a new and better admissible cost function.
    - ▷ *A way to get a new and better heuristic function by composing known heuristic functions.*
  - Make use of the fact that computers can memorize lots of patterns.
  - Solutions to pre-stored patterns can be pre-computed.
  - This year-2002 result has a speed up factor of over the 2,000 compared to a year-1985 previous result.

Definitions

- $n^2 - 1$ **puzzle problem:**
  - **The numbers 1 through $n^2 - 1$ are arranged in a n by n square with one empty cell.**
    - ▷ ***Let*** $N = n^2 - 1$.
  - **Slide the tiles to a given goal position.**
- **15 puzzle:**
  - **May be invented in 1874 and was popular in 1880.**
  - **It looks like one can rearrange an arbitrary state into a given goal state.**
  - **Publicized and published by Sam Lloyd in January 1896.**
    - ▷ ***A prize of US$ 1,000 was offered to solve one "impossible", but seems to be feasible case.***
    - ▷ ***Note: average wage per hour for a worker is US$0.3.***
    - ▷ ***Page 235, Cyclopedia of Puzzles, 1914, Sam Lloyd***
- **Generalizations:**
  - $n{\cdot}m - 1$ **puzzle.**
  - **Puzzles of different shapes.**

Original offer





THE 14-15 PUZZLE IN PUZZLELAND

The older inhabitants of Puzzleland will remember how in the early seventies I drove the entire world crazy over a little box of movable blocks which became known as the "14-15 Puzzle." The fifteen blocks were arranged in the square box in regular order, only with the 14 and 15 reversed, as shown in the above illustration. The puzzle consisted in moving the blocks about, one at a time, so as to bring them back to the present position in every respect except that the error in the 14 and 15 must be corrected.

A prize of $1,000, which was offered for the first correct solution to the problem, has never been claimed, although there are thousands of persons who say they performed the required feat.

People became infatuated with the puzzle and ludicrous tales are told of shopkeepers who neglected to open their stores; of a distinguished clergyman who stood under a street lamp all through a wintry night trying to recall the way he had performed the feat. The mysterious feature of the puzzle is that no one seems to be able to recall the sequence of moves whereby they feel sure they succeeded in solving the puzzle. Pilots are said to have wrecked their ships, engineers rush their trains past stations and business generally became demoralized. A famous Baltimore editor tells how he went for his noon lunch and was discovered by his frantic staff long past midnight pushing little pieces of pie around on a plate! Farmers are known to have deserted their plows and I have taken one of such instances as an illustration for the sketch.

Several new problems developed from the original puzzle which are worth giving:

Second Problem—Start again with the blocks as in Fig. 1 and move them so as to get the numbers in regular order, but with the vacant square at upper left-hand corner instead of lower right-hand corner; see Fig. 2.

Third Problem—Start with Fig. 1, turn the box a quarter way round and so move the blocks that they will rest as in Fig. 3.

Fourth Problem—This is to move the pieces about until they form a "magic square," so that the numbers will add up thirty in ten different directions.

Fig 2.

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15

Fig 3.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

The Picnic Puzzle.

When they started off on the great annual picnic every wagon in town was pressed into service. Half way to the grounds ten wagons broke down, so it was necessary for each of the remaining wagons to carry one more person.

When they started for home it was discovered that fifteen more wagons were out of commission, so on the return trip there were three persons more in each wagon than when they started out in the morning.

Now who can tell how many people attended the great annual picnic?

SAM LOYD'S GREAT PICNIC PUZZLE

235

Page 235, Cyclopedia of Puzzles, 1914, Sam Lloyd
http://www.mathpuzzle.com/loyd/

15 puzzle

- **Rules:**
  - **15 tiles in a 4*4 square with numbers from 1 to 15.**
  - **One empty cell.**
  - **A tile can be slid horizontally or vertically into an empty cell.**
  - **From an initial position, slide the tiles into a goal position.**
    - ▷ ***Optimal version: using the fewest number of moves.***
- **Examples:**
  - **Initial position:**

10	8		12
3	7	6	2
1	14	4	11
15	13	9	5

  - **Goal position:**

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

15 Puzzle — State Space

- **State space is divided into two disjoint subsets of even and odd permutations [Johnson & Story 1879].**
  - **Treat a board into a permutation by appending non-empty cells in the rows from left to right and from top to bottom.**
  - f_1 **is number of inversions in a permutation** $\pi_1\pi_2\cdots\pi_N$ **where an inversion is a distinct pair** $\pi_i > \pi_j$ **such that** $i < j$**.**
    - ▷ ***Let*** $inv(i,j) = 1$ ***if*** $\pi_i > \pi_j$ ***and*** $i < j$***; otherwise, it is*** 0***.***
    - ▷ $f_1 = \sum_{\forall i,j} inv(i,j)$***.***
    - ▷ ***Example: the permutation 10,8,12,3,7,6,2,1,14,4,11,15,13,9,5 has 9+7+9+2+5+4+1+0+5+0+2+3+2+1+0 = 50 inversions.***
  - f_2 **is the row number, i.e., 1, 2, 3, or 4, of the empty cell.**
  - $f = f_1 + f_2$**.**
  - **Board parity**
    - ▷ ***Even parity: one whose*** f ***value is even.***
    - ▷ ***Odd parity: one whose*** f ***value is odd.***

15 Puzzle — Properties 1 and 2

- **Property 1: The parity of a board is either even or odd.**
- **Property 2: There exists some boards with even parity and some other boards with odd parity.**
  - **There is a board with an even parity.**
    - ▷ *The goal position:*

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

    - ▷ $f_1 = 0$ *and* $f_2 = 4$.
  - **There is a board with an odd parity.**
    - ▷

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

    - ▷ $f_1 = 1$ *and* $f_2 = 4$.
  - **The above two form the cash-prize challenge posed by Sam Lloyd in 1914.**

15 Puzzle — Properties 3 and 4

- **Property 3: Slide a tile never change the parity of a 15-puzzle board.**
  - **A proof sketch is given in the next slide.**
- **Property 4: Given two boards with the same parity, we can obtain one from the other by sliding tides.**
  - **Proof is omitted.**
  - **Note: it suffices to pick a fixed goal position for the even/odd permutations. Then prove every other permutation of the same parity can be slid into this picked goal position.**
    - ▷ ***If A can be slid into G, and B can be slid into G, then A can be slid into B, and vice versa.***

Proof sketch of Property 3

- **Slide a tile horizontally does not change the parity.**
- **Slide a tile vertically:**
  - **Change the parity of f_2, i.e., row number of the empty cell.**
  - **Change the value of f_1, i.e., the number of inversions by**
    - $+3$
    - $+1$
    - -1
    - -3
  - **Example: when "a" is slid down**
    - *only the relative order of "a", "b" , "c" and "d" are changed*
    - *analyze the 4 cases according to the rank of "a" in "a", "b" , "c" and "d".*

*	*	*	*
*	a	b	c
d		*	*
*	*	*	*

Generalizations (1/2)

- **Properties discussed here work for $n \times m - 1$ puzzles where n and m are both at least 2 [Johnson and Story 1879].**
  - **n is the number of rows and m is the number of columns.**
- **When n or m is 1, then obviously there is no way to get a different permutation by sliding from a starting permutation. Hence the parity argument does not work.**
- **The above parity arguments work for $n > 1$, $m > 1$ and m is even.**
  - **Take the following size-m sub-permutation ended with the empty square: $a_1, a_2, \ldots, a_m$.**
  - **After sliding a_1 down, the corresponding sub-permutation becomes $a_2, a_3, \ldots, a_m, a_1$.**
  - **Let P be the original permutation, and P' be the resulting permutation by sliding down a_1.**
  - **When a_1 is the j, $1 \leq j \leq m$, largest in the sub-permutation, then $f_1(P') = f_1(P) - m + 1 + 2 \cdot (j - 1) = f_1(P) - m - 1 + 2 \cdot j$.**
  - **$abs(-m - 1 + 2 \cdot j)$ is odd when m is even.**

Generalizations (2/2)

- **When $n > 1$, $m > 1$ and m is odd.**
  - $f_1(P)$ is even when m is odd.
  - Let $f'_2(P) = 0$.
  - Then the proof still works.
- **To have a uniform definition, the proof is more complex, but doable.**
  - Revised definitions
    - ▷ *Insert the empty square numbered $n \times m$, namely largest one in the permutation, into the permutation and call it f''_1.*
    - ▷ *f''_2 is the Manhattan distance between the empty square and the right lower corner.*
  - Every sliding, horizontal and vertical, changes the parity of f''_2.
  - Horizontal sliding always increases f''_1 by 1.
  - Vertical sliding also changes the parity of f''_1.
  - Hence the parity argument also works here.
- **Other sizes or types of sliding piece puzzles are challenging and worth researching individually.**
  - Ref: Sliding Piece Puzzles, Edward Hordern, 1986, Oxford University Press, ISBN 0-19-853204-0

Core of past algorithms

- **Using DEC 2060 a 1-MIPS machine: solves several random instances of the 15 puzzle problem within 30 CPU minutes in 1985.**
- **Using Iterative-deepening A*.**
- **Using the Manhattan distance heuristic as an estimation of the remaining cost.**
  - **Suppose a tile is currently at (i, j) and its goal is at (i', j'), then**
    - *the Manhattan distance for this tile is $|i - i'| + |j - j'|$.*
  - **The Manhattan distance between a board and a goal board is the sum of the Manhattan distance of all the tiles.**
- **Manhattan distance is a lower bound on the number of slides needed to reach the goal position.**
  - **It is admissible.**
  - **Not good enough in terms of speed and space for solving the 24 puzzle problem.**

Non-additive pattern databases

- **Intuition: do not measure the distance of one tile at a time.**
  - **Pattern database: measure the collective distance of a pattern, i.e., a group of tiles, at a time.**
- **Complications.**
  - **The tiles get in each other's way.**
  - **Sliding a tile to reach its goal destination may make the other tiles that are already in their destinations to move away.**
  - **A form of interaction is called linear conflict:**
    - ▷ ***To flip two adjacent tiles needs more than 2 moves.***
    - ▷ ***In addition, sliding tiles other than the two adjacent tiles to be flipped is also needed in order to flip them.***

Example: Linear conflict

- **The sum of Manhattan distance between the board on the left and the goal board on the right is 4.**

1	2	3	4
5	6	7	8
9	12	10	11
13	14	15	

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

- **However it takes much more than 4 slides to reach the goal.**

1	2	3	4
5	6	7	8
9	12	10	11
13	14	15	

$\Longrightarrow \cdots \Longrightarrow$

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Fringe (1/2)

- **A fringe is the arrangement of a subset of tiles, and may include the empty cell, by treating tiles not selected don't-care.**
  - **Don't-cared tiles are indistinguishable within themselves.**
  - **The subset of tiles selected is called a pattern.**
  - **Example: original**

1	2	**4**	
3	**8**	5	**12**
6	**13**	7	**15**
9	10	**14**	11

fringe

$*$	$*$	**4**	
$*$	**8**	$*$	**12**
$*$	**13**	$*$	**15**
$*$	$*$	**14**	$*$

- **Notations for specifying a pattern.**
  - **"$*$" means don't-care.**
  - **We need to know the whereabout of the empty cell no matter it is selected or not.**
    - ▷ *An empty space means a selected empty cell.*
    - ▷ *"$\heartsuit$" means an unselected empty cell.*

Fringe (2/2)

- **Example:**

$*$	$*$	**4**	
$*$	**8**	$*$	**12**
$*$	**13**	$*$	**15**
$*$	$*$	**14**	$*$

- **In this example, there are 7 selected tiles, including the empty cell.**
  - **There are** $16!/9! = 57,657,600$ **possible fringe arrangements which is called the pattern size.**
- **The goal fringe arrangement for the selected subset of tiles:**

$*$	$*$	$*$	**4**
$*$	$*$	$*$	**8**
$*$	$*$	$*$	**12**
13	**14**	**15**	

Solving a fringe arrangement

- **For each fringe arrangement, pre-compute the minimum number of moves needed to make it into the goal fringe arrangement.**
  - **This is called the fringe number for the given fringe arrangement.**
  - **There are many possible ways to solve this problem since the pattern size is small enough to fit into the main memory.**
    - ▷ *Sample solution 1: Using the original Manhattan distance heuristic to solve this smaller problem.*
    - ▷ *Sample solution 2: BFS.*
- **Remark: It is not necessary to include everything into the fringe selections to solve a puzzle. In a $n^2 - 1$ puzzle, there are $n^2 - 1$ tiles and 1 empty square. It suffices to not picking any one in the fringe.**
  - **Using a generalized pigeon hole principle, it can be argued that once you put $n^2 - 1$ selections into the right locations, the one left out is sure to be in the right location.**

Comments on pattern size

- **Pro's.**
  - **Pattern with a larger size is better in terms of having a larger fringe number.**
  - **A larger fringe number usually means better estimation, i.e., closer to the goal fringe arrangement.**
- **Con's.**
  - **Pattern with a larger size means consuming lots of memory to memorize these arrangements.**
  - **Pattern with a larger size also means consuming lots of time in constructing these arrangements.**
    - ▷ ***Depend on your resource, pick the right pattern size.***

Usage of fringe numbers (1/2)

- **Divide and conquer.**
  - **Reduce a 15-puzzle problem into a 8-puzzle one.**
  - **Solution =**
    - ▷ ***First reach a goal fringe arrangement consisted of the first row and column.***
    - ▷ ***Then solve the 8-puzzle problem without using the fringe tiles.***
    - ▷ ***Finally Combining these two partial solutions to form a solution for the 15-puzzle problem.***

♡	$*$	$*$	4
13	$*$	3	$*$
$*$	9	5	$*$
$*$	2	$*$	1

$\Longrightarrow$

1	2	3	4
5	$*$	♡	$*$
9	$*$	$*$	$*$
13	$*$	$*$	$*$

  - **May not be optimal.**
- **Divide and conquer may not be working because often times you cannot combine two sub-solutions to form the final optimal solution easily.**
  - **In solving the second half, you may affect tiles that have reached the goal destinations in the first half.**
  - **The two partial solutions may not be disjoint.**

Usage of fringe numbers (2/2)

- **New heuristic function $h()$ for IDA*: using the fringe number as the new lower bound estimation.**
  - **The fringe number is a lower bound on the remaining cost.**
    - ▷ *It is admissible.*
    - ▷ *Q: how to prove it is admissible?*
- **How to find better patterns for fringes?**
  - **Large pattern require more space to store and more time to compute.**
  - **Can we combine smaller patterns to form bigger patterns?**
    - ▷ *They are not **disjoint**.*
    - ▷ *May be overlapping physically.*
    - ▷ *May be overlapping in solutions.*

More than one patterns

- **Can have many different patterns that may have some overlaps:**

$*$	$*$	**3**	$*$
$*$	$*$	**7**	$*$
9	**10**	**11**	**12**
$*$	$*$	**15**	♡

1	**2**	**3**	**4**
5	$*$	$*$	$*$
9	$*$	$*$	$*$
13	$*$	$*$	♡

  - **Cannot use the divide and conquer approach anymore for some of the patterns.**
- **If you have many different pattern databases P_1, P_2, P_3, $\ldots$**
  - **The heuristics or patterns may not be disjoint.**
    - ▷ ***Solving tiles in one pattern may help/hurt solving tiles in another pattern even if they have no common cells.***
  - **The heuristic function we can use is**

$$h(P_1, P_2, P_3, \ldots) = \max\{h(P_1), h(P_2), h(P_3), \ldots\}.$$

Problems with multiple patterns (1/2)

- **If you have many different pattern databases P_1, P_2, P_3, $\ldots$**
  - **It is better to have**
    - $h(P_1, P_2, P_3, \ldots) = h(P_1) + h(P_2) + h(P_3) + \cdots,$

    instead of
    - $h(P_1, P_2, P_3, \ldots) = \max\{h(P_1), h(P_2), h(P_3), \ldots\}.$
  - **A larger $h()$ means a better performance for A*.**
- **Key problem: how to make sure $h()$ is admissible through the composition process?**

Problems with multiple patterns (2/2)

- **Making the heuristics and the patterns disjoint and beyond.**
  - **If the patterns are not disjoint, then we cannot add them together.**
    - ▷ *Divide the board into several disjoint regions.*
  - **Though patterns are disjoint, their costs are not disjoint.**
    - ▷ *Some moves are counted more than once.*
- **Q: Why can we add the Manhattan distance of all tiles together to form a heuristic function?**
  - **We add 15 1-cell patterns together to form a better heuristic function.**
  - **What are the properties of these patterns so that they can be added together?**

Key observations (1/2)

- **Partition the board into disjoint regions.**
  - **Using the tiles in a region of the goal arrangement as a pattern.**
- **Examples:**
  -
A	A	A	A
A	A	A	A
B	B	B	B
B	B	B	B

  -
A	A	B	B
A	A	B	B
A	A	B	B
A	A	B	B

- **Can also divide the board into more than 2 disjoint patterns.**
  -
A	A	A	B
A	A	B	B
C	A	C	B
C	C	C	B

Key observations (2/2)

- **For each region, solve the problem optimally and then count the moves that are made only by tiles in this region.**
  - **Note: if the empty cell is selected, we do not count the moves of the empty cell.**
  - **The "fringe" number for an arrangement is the minimum number of slides made on tiles in this region.**
  - **It is now possible to add fringe numbers of all disjoint regions together to form a composite fringe number.**
    - ▷ *Q: How to prove this?*
- **For the Manhattan distance heuristic:**
  - **Each pattern is a tile.**
  - **They are disjoint.**
    - ▷ *They only count the number of slides made by each tile.*
  - **Thus they can be added together to form a heuristic function.**

Disjoint patterns

- A heuristic function $f()$ is disjoint with respect to two patterns P_1 and P_2 if
  - P_1 and P_2 have no common cells.
    - ▷ *Example:*

1	2	3	4
5	$*$	$*$	$*$
9	$*$	$*$	$*$
13	$*$	$*$	♡

$*$	$*$	$*$	$*$
$*$	6	7	8
$*$	9	10	11
$*$	14	15	

  - The solutions corresponding to $f(P_1)$ and $f(P_2)$ do not interfere each other.
    - ▷ *The above example does interfere each other.*
- Then $f(P_1) + f(P_2)$ is admissible if
  (1) $f()$ is disjoint with respect to P_1 and P_2 and
  (2) both $f(P_1)$ and $f(P_2)$ are admissible.
  - Q: How to prove this?

Revised fringe number

- **Fringe number: for each fringe arrangement, the minimum number of moves needed to make it into the goal fringe arrangement.**
  - **Given a fringe arrangement P, let $f(P)$ be its fringe number.**
- **Revised fringe number: for each fringe arrangement during the course of making a sequence of moves to the goal fringe arrangement, the minimum number of fringe-only moves in the sequence of moves.**
  - **Given a fringe arrangement P, let $f'(P)$ be its revised fringe number.**
- **Given two patterns P_1 and P_2 without overlapping cells, then**
  - **$f(P_1)$ and $f'(P_1)$ are both admissible.**
  - **$f(P_2)$ and $f'(P_2)$ are both admissible.**
  - **$f(P_1) + f(P_2)$ is not admissible.**
  - **$f'(P_1) + f'(P_2)$ is admissible.**
- **Note: the Manhattan distance of a 1-cell pattern is a lower bound of its revised fringe number.**

Comments

- A special form of divide and conquer with additional properties.
- Spaces required by patterns must be within the main memory.
- Each pattern must be able to be solved optimally by "primitive" methods.
- It is better to put near-by tiles together to better deal with the conflicting problem.
- It is now possible to design a better admissible heuristic function f by composing two simple admissible heuristic functions f_1 and f_2.
  - Let f_1' be the function that does not count moves of tiles not in its region when computing f_1.
    - ▷ $f_1'(x) \leq f_1(x)$
  - Let f_2' be the function that does not count moves of tiles not in its region when computing f_2.
    - ▷ $f_2'(x) \leq f_2(x)$
  - Let $f = f_1' + f_2'$.
    - ▷ ***Hopefully,*** $f(x) > f_1(x)$ ***and*** $f(x) > f_2(x)$.

Performance

- **Running on a 440-MHZ Sun Ultra 10 workstation.**
  - **SPECint = 1.0 (1 MIPS) in 1985.**
  - **SPECint = 17.9 in 2002.**
- **Solves the 15 puzzle problem that is more than 2,000 times faster than the previous result by using the Manhattan distance heuristic.**
  - **2,000 * 17.9 times faster in wall clock time.**
- **Solves the 24-puzzle problem**
  - **An average of two days per problem instance.**
  - **Generates 2,110,000 nodes per second.**
  - **The average solution length was 100.78 moves.**
  - **The maximum solution length was 114 moves.**
  - **Prediction: using the Manhattan distance heuristic, it would take an average of about 50,000 years to solve a problem instance.**
    - ▷ ***The average Manhattan distance is 76.078 moves.***
    - ▷ ***The average value for the disjoint database heuristic is 81.607 moves, which gives a tighter bound.***
    - ▷ ***The improvement of heuristic is only 7.27%, but the speed is 2,000 times faster.***

Other heuristics (1/3)

- **One of the main drawbacks of using the disjoint heuristics is that it does not capture interactions between tiles in different regions.**
- **2-tile pattern database:**
  - **For each pair of tiles, and for each pair of possible destinations, compute the optimal solution, i.e., minimum number of all moves made by these 2 tiles, for this pair of tiles to both move to their destinations.**
    - ▷ ***Example: two tiles i and j can be at locations $loc(i)$ and $loc(j)$ and want to move to destinations $des(i)$ and $des(j)$.***
    - ▷ ***Usually, the destination is fixed for a tile in a puzzle.***
    - ▷ ***The shortest moves to move both tiles i and j to their destinations is called their pairwise distance.***
  - **Complexities:**
    - ▷ ***Assume the destinations are fixed in a puzzle, then we need to compute $O(n^4)$ pairwise distance for an n^2-1 puzzle since the number of possible locations of tile i, $loc(i)$, is $O(n^2)$.***
    - ▷ ***For $n = 4$, $n^4 = 256$.***
    - ▷ ***For $n = 5$, $n^4 = 625$.***

Other heuristics (2/3)

- **It is usually the case that the pairwise distance of 2 tiles x and y is much larger than the sum of the Manhattan distances of x and y.**
  - **The pairwise distance is at least the sum of the Manhattan distances.**
  - **Q: How to prove this?**
- **For a given board, partition the board into a collection of 2-tiles so that the sum of cost is maximized.**
  - **This is called the maximum sum of pairwise distance.**
  - **For partitioning the board, we mean to find eight 2-titles so that they cover all tiles, including the empty cell.**
  - **This new cost estimation function is admissible.**
    - ▷ ***Q: How to prove this?***

Other heuristics (3/3)

- **Finding a maximum sum of pairwise distance can be done using a maximum weighted perfect matching.**
  - **Build a complete graph with the tiles being the vertices.**
  - **The edge cost is the pairwise distance between these two tiles.**
  - **Try to find a perfect matching with the sum of edge costs being the largest possible.**
  - **Algorithm runs in $O(\sqrt{n} \cdot m)$ time is known where n is the number of vertices and m is the number of edges.**
    - ▷ ***S. Micali and V.V. Vazirani, "An $O(\sqrt{|V|} \cdot |E|)$ algorithm for finding maximum matching in general graphs", Proc. 21st IEEE Symp. Foundations of Computer Science, pp. 17-27, 1980.***
    - ▷ ***Faster algorithms are known since the input is a complete graph.***

Comments

- **The Manhattan distance is a partition into 1-tile patterns.**
- **For 2-tile patterns:**
  - **Faster approximation algorithms for finding maximum perfect matchings on complete graphs are known.**
  - **The cost for exhaustive enumeration is**
    - ▷ $$\binom{16}{2}\binom{14}{2}\cdots\binom{4}{2}\binom{2}{2}/8!$$
    - ▷ $= 16!/(2^8 \cdot 8!) = 2,027,025$
  - **The above formula gives the number of distinct number of ways to partition 16 titles into 8 pairs.**
- **Can also build 3-tile databases, but the corresponding 3-D matching problem for partitioning is NP-hard.**
- **Requires much less memory than that of the the fringe method.**
- **Some kinds of bootstrapping: solving smaller problems using primitive methods, and then using these results to solve larger problems.**

What else can be done?

- **Looks like some kinds of two-stage search.**
  - First stage searching means building pre-computed results, e.g., patterns.
  - Second stage searching meets the pre-computed results if found.
- **Better way of partitioning.**
- **Is it possible to generalize this result to other problem domains?**
- **How to decide the amount of time used in searching and the amount of time used in retrieving pre-computed knowledge?**
  - Memorize vs Compute

References and further readings

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